

Advanced Game Theory

# Solution concepts

Lecture 2

Minimax

Rationalizability

Bayesian Nash equilibrium

Correlated equilibrium

Olivier Gossner

# From information to solution concepts

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## Lecture 1

Partitions

Knowledge and common knowledge

Conditional beliefs

## Lecture 2

Predictions of play

Beliefs and equilibrium

Role of information

How does information enter these solution concepts?

# Four lectures

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- 1 Knowledge and agreement**  
Partitions; common knowledge; Aumann's theorem
- 2 Solution concepts**  
Minimax; rationalizability; Bayesian games; correlated equilibrium
- 3 Comparing and choosing information**  
Blackwell; strategic value; information design
- 4 Higher-order beliefs and universal representation**  
Email game; universal type space; interim rationalizability

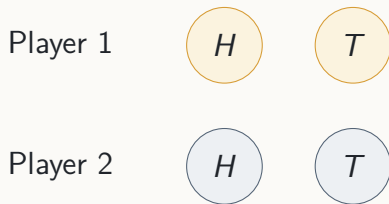
## Lecture 2: Minimax

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- 1 Minimax**  
Guarantees; mixing; a separation argument
- 2 Rationalizability**  
Rational choice; iterated deletion; largest fixed point
- 3 Bayesian games**  
Types; contingent plans; best responses at each type
- 4 Correlated equilibrium**  
Signals without payoff news; obedience; revelation

# Matching pennies

Both players choose heads ( $H$ ) or tails ( $T$ ) simultaneously.



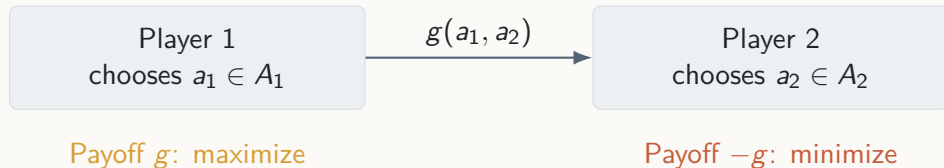
|     | $H$     | $T$     |
|-----|---------|---------|
| $H$ | $1, -1$ | $-1, 1$ |
| $T$ | $-1, 1$ | $1, -1$ |

Player 1: row. Player 2: column.

Player 1 wins when the faces match; Player 2 wins when they differ.

## A zero-sum game

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How much can Player 1 guarantee? How much can Player 1 defend?

## Guarantee: Player 1 chooses first

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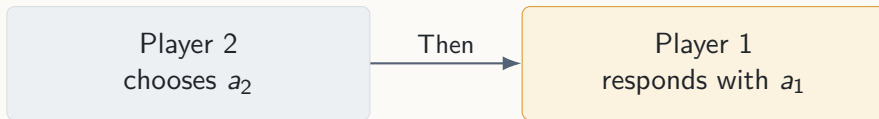
Player 1 can guarantee  $w$  if

$$\exists a_1 \quad \forall a_2 : \quad g(a_1, a_2) \geq w.$$

Best pure-strategy guarantee:  $\underline{v}^P = \max_{a_1 \in A_1} \min_{a_2 \in A_2} g(a_1, a_2).$

## Defend: Player 1 responds to Player 2

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Player 1 can defend  $w$  if

$$\forall a_2 \quad \exists a_1 : \quad g(a_1, a_2) \geq w.$$

$$\text{Best pure-strategy defense: } \bar{v}^P = \min_{a_2 \in A_2} \max_{a_1 \in A_1} g(a_1, a_2).$$

## Matching pennies: a gap with pure strategies

|     | $H$ | $T$ |
|-----|-----|-----|
| $H$ | 1   | -1  |
| $T$ | -1  | 1   |

Player 1's payoffs.

### Guarantee

Player 2 chooses the other face.

$$\underline{v}^P = -1.$$

### Defend

Player 1 matches Player 2's face.

$$\bar{v}^P = 1.$$

# It is always easier for Player 1 to defend

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## Guarantee

$$\exists a_1 \forall a_2 : g(a_1, a_2) \geq w.$$

## Defend

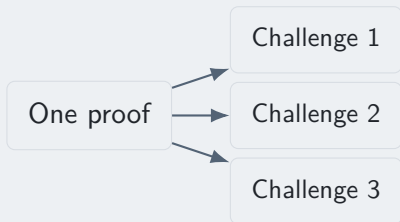
$$\forall a_2 \exists a_1 : g(a_1, a_2) \geq w.$$

A guaranteeing strategy works against every action of Player 2.

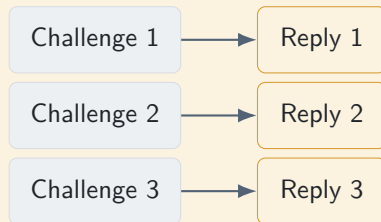
$$\underline{v}^P \leq \overline{v}^P$$

# Proofs and counterexamples: a quantifier-order analogy

## Guarantee: find a proof



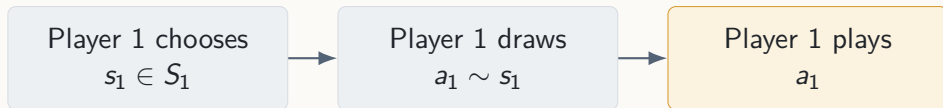
## Defend: beat counterexamples



## Mixing: Player 1 chooses a distribution

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$$S_i = \Delta(A_i), \quad g(s_1, s_2) = \sum_{a_1, a_2} s_1(a_1) s_2(a_2) g(a_1, a_2).$$



To defend, Player 1 responds to Player 2's distribution before Player 2 draws an action.

# Guarantee and defend with mixed strategies

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## Player 1's guarantee

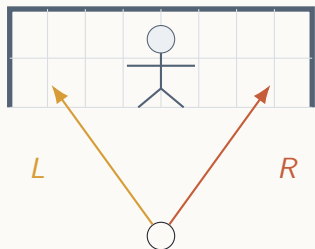
$$\underline{v} = \max_{s_1} \min_{s_2} g(s_1, s_2).$$

## Player 1's defense

$$\overline{v} = \min_{s_2} \max_{s_1} g(s_1, s_2).$$

$$\underline{v} \leq \overline{v} \quad \text{Does mixing close the gap?}$$

## Penalty kicks: 80% versus 60%

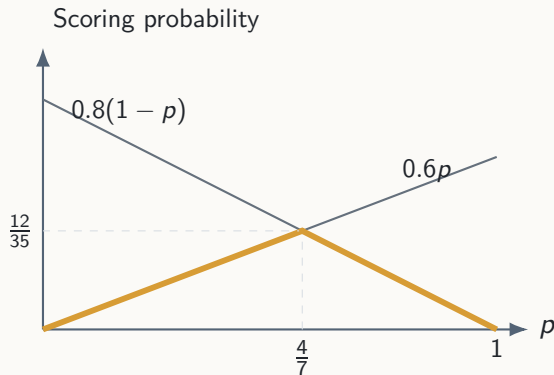


|          | <i>l</i> | <i>r</i> |
|----------|----------|----------|
| <i>L</i> | 0%       | 80%      |
| <i>R</i> | 60%      | 0%       |

Player 1's scoring probability

Player 1 (shooter): row. Player 2 (goalkeeper): column.

## Player 1's guarantee



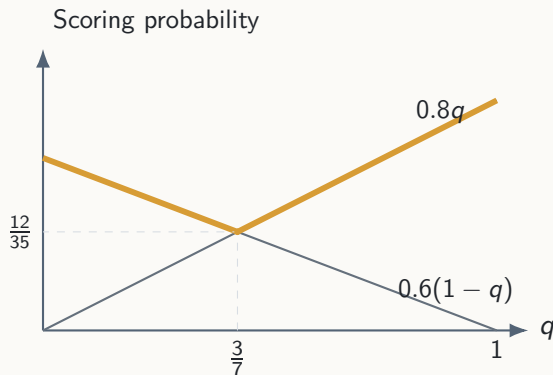
$p = \Pr(\text{Player 1 plays } R).$

Against  $l$ :  $0.6p$ .

Against  $r$ :  $0.8(1 - p)$ .

Player 1 uses  $p = \frac{4}{7}$  and guarantees at least  $\frac{12}{35} \simeq 34.3\%$ .

## Player 1's defense



$q = \Pr(\text{Player 2 plays } r).$

Given Player 2's mixture  $q$ ,  
Player 1 chooses the better kick:

$$\max\{0.8q, 0.6(1-q)\}.$$

The worst  $q$  is  $\frac{3}{7}$ .

Against every mixture of Player 2, Player 1 can score at least  $\frac{12}{35} \simeq 34.3\%$ .

## The gap closes

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### Player 1's optimal mixture

$$p^* = \Pr(R) = \frac{4}{7}.$$

Guarantees at least  $\frac{12}{35}$  against Player 2.

### Player 2's optimal mixture

$$q^* = \Pr(r) = \frac{3}{7}.$$

Limits Player 1's payoff to  $\frac{12}{35}$ .

Player 1's guarantee = Player 1's defense:  $\underline{v} = \overline{v} = \frac{12}{35}$ .

Why does Player 1 use the less effective kick more often?

# The minimax theorem

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von Neumann (1928)   Finite action sets; mixed strategies.

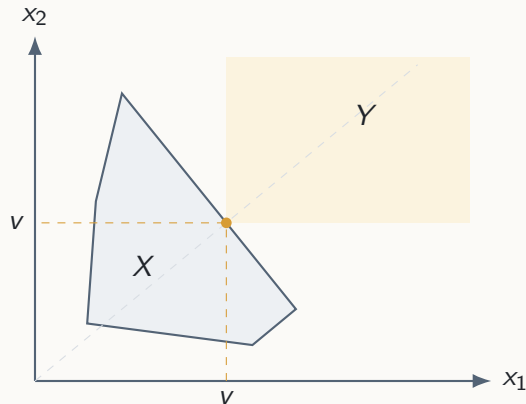
$$\underbrace{\max_{s_1 \in S_1} \min_{s_2 \in S_2} g(s_1, s_2)}_{\text{guarantee}} = \underbrace{\min_{s_2 \in S_2} \max_{s_1 \in S_1} g(s_1, s_2)}_{\text{defend}} = v.$$

Optimal strategies  $s_1^*, s_2^*$  satisfy

$$g(s_1, s_2^*) \leq v \leq g(s_1^*, s_2) \quad \text{for every } s_1, s_2.$$

$(s_1^*, s_2^*)$  is a Nash equilibrium.

## Proof: payoff vectors and an unattainable region



One coordinate per action  $j \in A_2$  of Player 2:

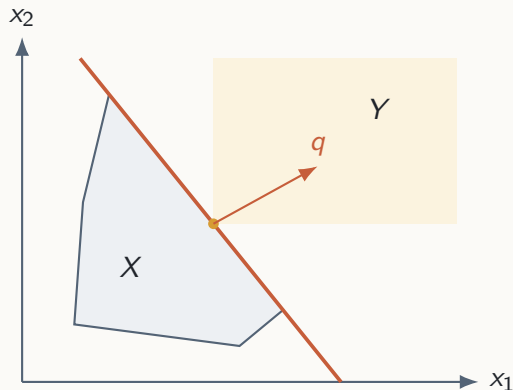
$$X = \{(g(s_1, j))_j : s_1 \in S_1\}.$$

$$v = \max_{x \in X} \min_j x_j.$$

$$Y = \{y : y_j > v \text{ for all } j\}.$$

$X$  is compact and convex;  $X \cap Y = \emptyset$ .

## Proof: the separator is Player 2's mixture



Separation gives  $q \neq 0$  with

$$q \cdot x \leq q \cdot y \quad (x \in X, y \in Y).$$

Let each coordinate of  $y$  grow:  $q_j \geq 0$ .

Normalize:  $\sum_j q_j = 1$ .

Let  $y \downarrow (v, \dots, v)$ :

$$g(s_1, q) = q \cdot x \leq v.$$

$$\bar{v} \leq v = \underline{v} \quad \text{and hence equality.}$$

## Lecture 2: Rationalizability

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Types; contingent plans; best responses at each type
- 4 Correlated equilibrium**  
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# I. What does rational mean?

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A finite game, with payoffs  $g_i(a_i, a_{-i})$ .

## Undominated

No mixed strategy strictly dominates the chosen action.



## Best response

Optimal for some conjecture about opponents' actions.

# Strict dominance

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Opponents' profiles:  $a_{-i} \in A_{-i} = \prod_{j \neq i} A_j$ .

A mixture  $s_i \in \Delta(A_i)$  **strictly dominates** an action  $a_i \in A_i$  if

$$g_i(s_i, a_{-i}) > g_i(a_i, a_{-i})$$

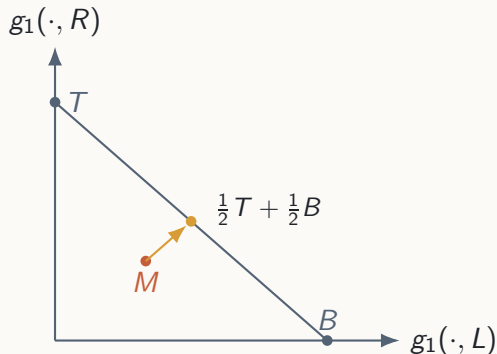
for every  $a_{-i} \in A_{-i}$ .

An action is **strictly dominated** if such a mixture exists.

## A mixture can dominate a pure action

|     | $L$ | $R$ |
|-----|-----|-----|
| $T$ | 0   | 3   |
| $M$ | 1   | 1   |
| $B$ | 3   | 0   |

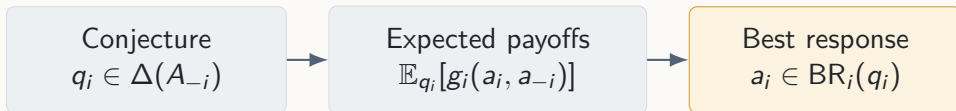
Only Player 1's payoffs.



$\frac{1}{2}T + \frac{1}{2}B$  yields  $\frac{3}{2} > 1$  against both columns.

## Best responses: rational choice needs a conjecture

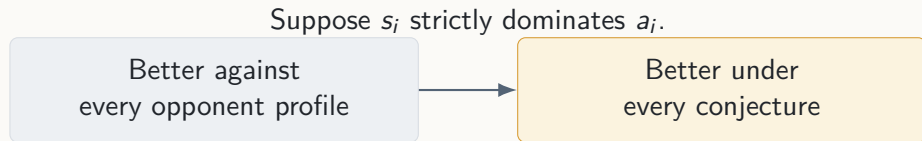
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$$BR_i(q_i) = \arg \max_{a_i \in A_i} \sum_{a_{-i}} q_i(a_{-i}) g_i(a_i, a_{-i}).$$

## Strict dominance rules out a best response

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$$\mathbb{E}_{q_i}[g_i(s_i, a_{-i})] > \mathbb{E}_{q_i}[g_i(a_i, a_{-i})] \quad \text{for every } q_i.$$

# Dominated iff never a best response

Strictly dominated by a mixture  $\iff$  never a best response.

## Never a best response

$$\begin{aligned} \forall q_i \quad \exists b_i : \\ \mathbb{E}_{q_i}[g_i(b_i, a_{-i})] \\ > \mathbb{E}_{q_i}[g_i(a_i, a_{-i})]. \end{aligned}$$

## One dominating mixture

$$\begin{aligned} \exists s_i \quad \forall a_{-i} : \\ g_i(s_i, a_{-i}) \\ > g_i(a_i, a_{-i}). \end{aligned}$$

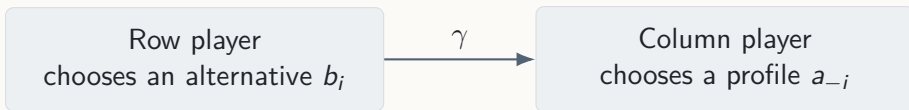
The minimax theorem turns local improvements into one uniform improvement.

## Proof: build a game of payoff improvements

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Fix the action  $a_i$ .

$$\gamma(b_i, a_{-i}) = g_i(b_i, a_{-i}) - g_i(a_i, a_{-i}).$$



If  $a_i$  is never a best response,

$$\max_{b_i} \mathbb{E}_{q_i}[\gamma(b_i, a_{-i})] > 0 \quad \text{for every } q_i.$$

Continuity and compactness give a strictly positive minimum.

## Proof: apply the minimax theorem

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$$\underbrace{\min_{q_i} \max_{b_i} \mathbb{E}_{q_i}[\gamma(b_i, a_{-i})]}_{\text{never a best response}} = \underbrace{\max_{s_i} \min_{a_{-i}} \gamma(s_i, a_{-i})}_{\text{one mixture works everywhere}} > 0.$$

Choose a maximizing  $s_i$ . Then

$$g_i(s_i, a_{-i}) - g_i(a_i, a_{-i}) > 0 \quad \text{for every } a_{-i}.$$

$s_i$  strictly dominates  $a_i$ .

## II. Iterating rationality

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I am rational; I know you are rational; I know you know. . .

### Iterated deletion

Delete strictly dominated actions.

Repeat in the reduced game.

### Largest fixed point

Actions justify one another.

Take the largest such family.

Do the two approaches select the same actions?

## Iterated deletion: two rounds of reasoning

|     | $L$  | $R$  |
|-----|------|------|
| $T$ | 3, 2 | 2, 1 |
| $B$ | 1, 0 | 0, 3 |

### Rationality

$T$  strictly dominates  $B$   
Player 1 chooses  $T$



### Knowledge of rationality

Player 2 expects  $T$   
Player 2 chooses  $L$

The only surviving profile is  $(T, L)$ .

## Iterated deletion: keep undominated actions

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$$R_i^0 = A_i, \quad R_{-i}^n = \prod_{j \neq i} R_j^n,$$

$$R_i^{n+1} = \{a_i \in R_i^n : a_i \text{ is undominated against } R_{-i}^n\}.$$

$$R_i = \bigcap_{n \geq 0} R_i^n \quad (\text{surviving actions}).$$

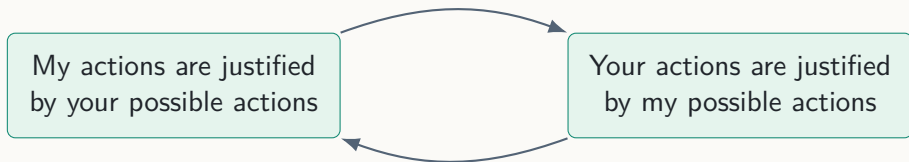


Finite action sets: the decreasing sequence eventually stabilizes.

## Largest fixed point: mutual justification

Let  $B_{-i} = \prod_{j \neq i} B_j$ . A family  $(B_i)_i$  is jointly rationalizable if

$$\forall i, \forall b_i \in B_i, \exists q_i \in \Delta(B_{-i}) : b_i \in \text{BR}_i(q_i).$$



$$\text{Largest family: } \tilde{R}_i = \bigcup_{(B_j) \text{ jointly rationalizable}} B_i.$$

Fixed point: taking best responses adds no new actions.

## Iterated deletion and the largest fixed point coincide

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**Iterated deletion**

Surviving actions

$R_i$



**Largest fixed point**

Rationalizable actions

$\tilde{R}_i$

$R_i = \tilde{R}_i$  for every player  $i$ .

## Proof: the two inclusions

### Survivors are justified

Stop:  $R_i^{n+1} = R_i^n = R_i$ .

Part I: each survivor is justified by a conjecture on  $R_{-i}$ .

Maximality:  $R_i \subseteq \tilde{R}_i$ .

### Justified actions survive

Start:  $\tilde{R}_i \subseteq R_i^0$ .

If  $\tilde{R}_j \subseteq R_j^n$  for all  $j$ :  
conjectures on  $\tilde{R}_{-i}$  are on  $R_{-i}^n$ .

Part I gives  $\tilde{R}_i \subseteq R_i^{n+1}$ .

Induction:  $\tilde{R}_i \subseteq \bigcap_n R_i^n = R_i$ .

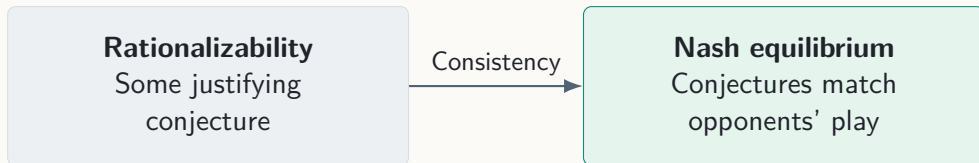
## Lecture 2: Bayesian games

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- 1 Minimax**  
Guarantees; mixing; a separation argument
- 2 Rationalizability**  
Rational choice; iterated deletion; largest fixed point
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Types; contingent plans; best responses at each type
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# Nash equilibrium: consistent conjectures

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$$g_i(s_i, s_{-i}) \geq g_i(s'_i, s_{-i}) \quad \text{for every } i, s'_i \in \Delta(A_i).$$

# Bayesian games: information and payoffs

## Information

States  $\omega \in \Omega$ .

Common prior  $p \in \Delta(\Omega)$ .

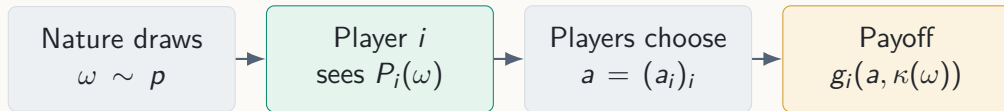
Player  $i$  observes the cell  $P_i(\omega)$  of a partition  $\mathcal{P}_i$ .

## Payoffs

Payoff state  $k = \kappa(\omega)$ .

Actions  $a_i \in A_i$ .

Payoffs  $g_i(a, \kappa(\omega))$ .



## Types are information cells

$$T_i = \mathcal{P}_i \quad t_i(\omega) = P_i(\omega).$$

$$t_i^a = \{\omega_1, \omega_2\}$$



Same information: same type

$$t_i^b = \{\omega_3, \omega_4\}$$

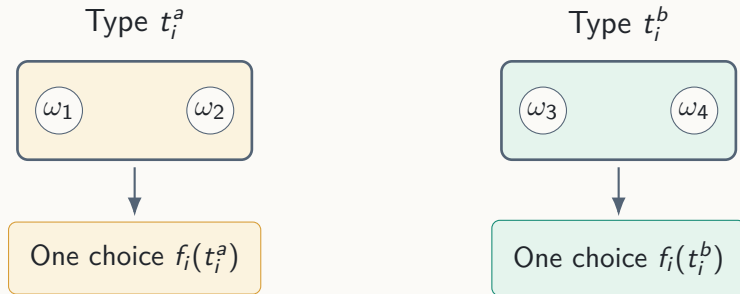


Same information: same type

$$p(t_i) = \sum_{\omega \in t_i} p(\omega).$$

## A strategy is a contingent plan

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Pure strategy:  $f_i : T_i \rightarrow A_i$ .

Behavioral strategy:  $\sigma_i : T_i \rightarrow \Delta(A_i)$ .

## Best response: one type at a time

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Bayes's rule:  $\omega \in t_i$ ,  $p(t_i) > 0$ .

$$p(\omega \mid t_i) = \frac{p(\omega)}{\sum_{\omega' \in t_i} p(\omega')}.$$

**Ex ante**

Best response among  
contingent plans



**At every type**

Best response whenever  
 $p(t_i) > 0$

# Why best responses separate by type

Fix opponents' contingent plans.

$$\underbrace{\mathbb{E}_p[g_i]}_{\text{ex ante}} = \sum_{\substack{t_i \in T_i \\ p(t_i) > 0}} p(t_i) \underbrace{\mathbb{E}_p[g_i \mid t_i]}_{\text{at type } t_i}.$$

## Profitable at one type

Change the action only at that type.

Its positive weight makes the ex ante payoff increase.

## Optimal at every type

Each type has its own choice.

No plan can improve the sum.

Actions at zero-probability types do not affect expected payoffs.

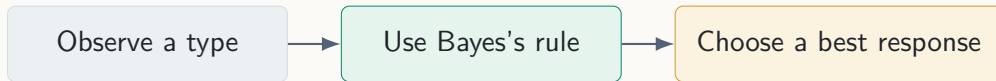
# Bayesian Nash equilibrium

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Nash equilibrium in contingent plans:

for every player  $i$  and every type  $t_i$  with positive probability,

$$\sigma_i(t_i) \in \arg \max_{s_i \in \Delta(A_i)} \mathbb{E}_p[g_i(s_i, \sigma_{-i}(t_{-i}(\omega)), \kappa(\omega)) \mid \omega \in t_i] .$$



With finite types and actions: Nash's theorem gives existence.

## Do you really want to defect?

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State  $L$

|     | $C$       | $D$       |
|-----|-----------|-----------|
| $C$ | $0, -2$   | $-10, -1$ |
| $D$ | $-1, -10$ | $-5, -5$  |

State  $R$

|     | $C$       | $D$       |
|-----|-----------|-----------|
| $C$ | $0, -2$   | $-10, -7$ |
| $D$ | $-1, -10$ | $-5, -11$ |

$$\Pr(L) = p, \quad \Pr(R) = 1 - p, \quad 0 < p < 1.$$

Player 1 chooses the row; Player 2 chooses the column.

## Case 1: nobody observes the state

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|     | $C$       | $D$           |
|-----|-----------|---------------|
| $C$ | $0, -2$   | $-10, 6p - 7$ |
| $D$ | $-1, -10$ | $-5, 6p - 11$ |

**Pure equilibrium  $(C, C)$**

$$p \leq \frac{5}{6}.$$

**Pure equilibrium  $(D, D)$**

$$p \geq \frac{1}{6}.$$

Two pure equilibria coexist for  $\frac{1}{6} \leq p \leq \frac{5}{6}$ .

## Case 2: only Player 2 observes the state

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A plan specifies an action in  $L$ , then an action in  $R$ .

$CC$

$CD$

$DC$

$DD$

**In state  $L$ : choose  $D$**

Against  $C$ :  $-1 > -2$ .

Against  $D$ :  $-5 > -10$ .

**In state  $R$ : choose  $C$**

Against  $C$ :  $-2 > -7$ .

Against  $D$ :  $-10 > -11$ .

**$DC$  is optimal at each type, whatever Player 1 does.**

## Player 1 responds to the informed plan

Against  $DC$ :

$$\mathbb{E}[g_1(C, DC)] = -10p, \quad \mathbb{E}[g_1(D, DC)] = -4p - 1.$$

$$C \text{ is optimal} \iff p \leq \frac{1}{6}.$$



At  $p = \frac{1}{6}$ , Player 1 may use any mixture of  $C$  and  $D$ .

## Lecture 2: Correlated equilibrium

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# Signals without payoff information

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## Fixed payoffs

One game:  $g_i(a)$ .

No uncertain payoff state.

## Private information

Joint signals  $(X_i)_i$ .

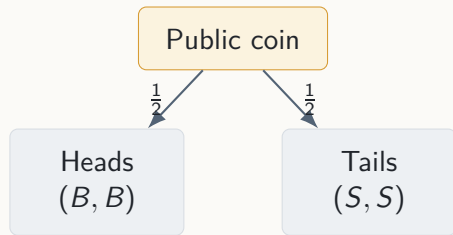
Player  $i$  observes only  $X_i$ .

Can signals that reveal nothing about payoffs still affect play?

## A public signal coordinates play

|     | $B$  | $S$  |
|-----|------|------|
| $B$ | 2, 1 | 0, 0 |
| $S$ | 0, 0 | 1, 2 |

Battle of the sexes.



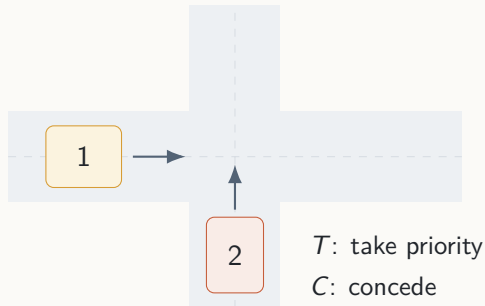
Following the coin is a Bayesian Nash equilibrium.

Both get  $\frac{3}{2}$  versus  $\frac{2}{3}$  in the mixed Nash equilibrium without signals.

## Can private signals do more?

|     | $C$  | $T$  |
|-----|------|------|
| $C$ | 6, 6 | 2, 7 |
| $T$ | 7, 2 | 0, 0 |

Chicken: concede or take priority.



Pure Nash equilibria:  $(T, C)$  and  $(C, T)$ .

## Each driver sees only their own traffic signal

|     | $R$           | $G$           |
|-----|---------------|---------------|
| $R$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $G$ | $\frac{1}{3}$ | 0             |

Joint signal distribution.



$C$ : concede



$T$ : take priority

A recommendation changes the conditional belief about the other driver.

## Best responses at each signal

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### Green: take priority

Other signal: red.

$$\mathbb{E}[T \mid G] = 7,$$

$$\mathbb{E}[C \mid G] = 6.$$

### Red: concede

Other signal: red or green, each with probability  $\frac{1}{2}$ .

$$\mathbb{E}[C \mid R] = \frac{1}{2}(6 + 2) = 4,$$

$$\mathbb{E}[T \mid R] = \frac{1}{2}(7 + 0) = \frac{7}{2}.$$

These plans form a Bayesian Nash equilibrium of the signal game.

## Private recommendations improve both payoffs

$$\mu(C, C) = \mu(C, T) = \mu(T, C) = \frac{1}{3}, \quad \mu(T, T) = 0.$$

|                                            | Player 1       | Player 2       |
|--------------------------------------------|----------------|----------------|
| Fair lottery over the pure Nash equilibria | $\frac{9}{2}$  | $\frac{9}{2}$  |
| Symmetric mixed Nash equilibrium           | $\frac{14}{3}$ | $\frac{14}{3}$ |
| Private traffic signals                    | 5              | 5              |

$$\mathbb{E}[g_i] = \frac{1}{3}(6 + 2 + 7) = 5.$$

# Obedience: correlated equilibrium

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Aumann (1974)

A mediator draws  $a \sim \mu \in \Delta(A)$ . Player  $i$ 's type is the recommendation  $a_i$ .

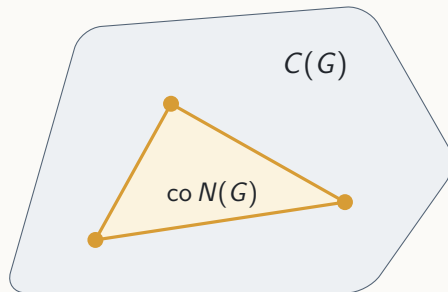
$$\sum_{a_{-i}} \mu(a_i, a_{-i}) [g_i(a_i, a_{-i}) - g_i(b_i, a_{-i})] \geq 0$$

for every  $i, a_i, b_i$ .

If  $\Pr(a_i) > 0$ :

$$\mathbb{E}[g_i(a_i, a_{-i}) \mid a_i] \geq \mathbb{E}[g_i(b_i, a_{-i}) \mid a_i].$$

# A convex set of obedient distributions



Schematic

$N(G)$ : Nash outcome distributions.

$C(G)$ : correlated-equilibrium distributions.

Obedience inequalities are linear in  $\mu$ .

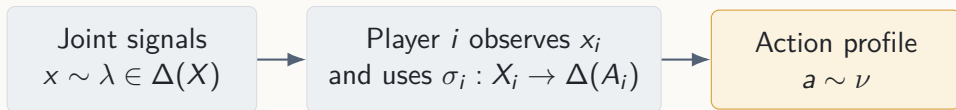
$C(G)$  is a compact convex polytope.

Nash distributions satisfy obedience;  
convexity also allows lotteries over them.

$$N(G) \subseteq \text{co } N(G) \subseteq C(G).$$

## A Bayesian equilibrium with arbitrary signals

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$$\nu(a) = \sum_x \lambda(x) \prod_i \sigma_i(a_i | x_i).$$

Fixed payoffs;  $\sigma = (\sigma_i)_i$  is a Bayesian Nash equilibrium.

# The revelation principle for correlated equilibrium

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Equilibrium with arbitrary private signals

$\iff$  an obedient distribution of action recommendations.



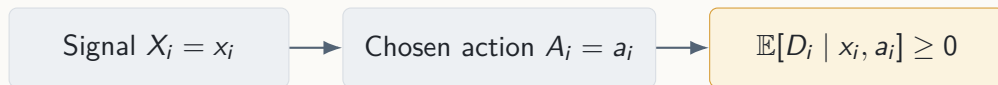
The action distribution  $\nu$  is preserved.

## Proof: keep the signal while checking a deviation

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Fix a player  $i$ , a recommended action  $a_i$ , and a deviation  $b_i$ .

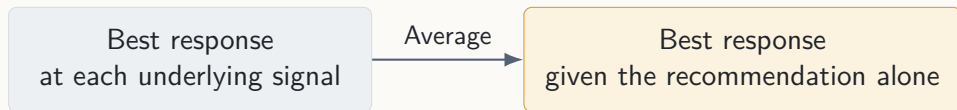
$$D_i = g_i(a_i, a_{-i}) - g_i(b_i, a_{-i}).$$



Every action used at a signal is a best response.  
Own randomization carries no extra information about opponents.

## Proof: average over the signals behind a recommendation

$$\mathbb{E}[D_i \mid A_i = a_i] = \sum_{x_i} \Pr(x_i \mid A_i = a_i) \underbrace{\mathbb{E}[D_i \mid X_i = x_i, A_i = a_i]}_{\geq 0} \geq 0.$$



$$\nu \in C(G).$$

Conversely: draw recommendations from  $\nu$ ; obedience gives best responses.

Lecture 1: averaging conditional expectations.

# Main references

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## Minimax

von Neumann (1928).

Chiappori, Levitt and Groseclose (2002).

## Rationalizability

Bernheim (1984); Pearce (1984).

Rationalizable strategic behavior.

## Bayesian games

Nash (1951), *Non-Cooperative Games*.

Harsanyi (1967–68): games with incomplete information.

## Correlated equilibrium

Aumann (1974), *Subjectivity and Correlation in Randomized Strategies*.

Maschler, Solan and Zamir (2013), *Game Theory*.

# Conclusion

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## Minimax

Guarantee equals defense

## Rationalizability

Common knowledge of rationality

## Bayesian games

Best responses, type by type

## Correlated equilibrium

Payoff-irrelevant signals can change play

Lecture 3: how do we compare information, value it, and choose it?

## Weak dominance: the order can matter

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|     | $L$  | $R$  |
|-----|------|------|
| $T$ | 1, 0 | 0, 2 |
| $M$ | 0, 2 | 2, 0 |
| $B$ | 1, 1 | 2, 1 |

### Delete $T$ first

Then  $L$  weakly dominates  $R$ .

Delete  $R$ , then  $M$ :  $(B, L)$ .

### Delete $M$ first

Then  $R$  weakly dominates  $L$ .

Delete  $L$ , then  $T$ :  $(B, R)$ .

## Conjectures: must opponents' actions be independent?

Player 1 chooses  $T, M, B$ ; Player 2 chooses  $L, R$ ; Player 3 chooses  $l, r$ .

Player 3 chooses  $l$

|     | $L$ | $R$ |
|-----|-----|-----|
| $T$ | -8  | 0   |
| $M$ | 0   | 0   |
| $B$ | -3  | -3  |

Player 3 chooses  $r$

|     | $L$ | $R$ |
|-----|-----|-----|
| $T$ | 0   | 0   |
| $M$ | 0   | -8  |
| $B$ | -3  | -3  |

Only Player 1's payoffs are displayed.

## The same marginals can justify different choices

### Independent conjecture

$$p = \Pr(L), q = \Pr(l).$$

$$\mathbb{E}[T] = -8pq,$$

$$\mathbb{E}[M] = -8(1-p)(1-q).$$

$$\max\{\mathbb{E}[T], \mathbb{E}[M]\} \geq -2 > -3.$$

**$B$  is never a best response.**

### Correlated conjecture

|     | $l$           | $r$           |
|-----|---------------|---------------|
| $L$ | $\frac{1}{2}$ | 0             |
| $R$ | 0             | $\frac{1}{2}$ |

$$\mathbb{E}[T] = \mathbb{E}[M] = -4.$$

**$B$  is the unique best response.**

At  $p = q = \frac{1}{2}$ , both conjectures have uniform marginals.

## The bound for independent conjectures

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Cauchy–Schwarz:

$$\sqrt{pq} + \sqrt{(1-p)(1-q)} \leq \sqrt{p + (1-p)} \sqrt{q + (1-q)} = 1.$$

Hence

$$\min\{\sqrt{pq}, \sqrt{(1-p)(1-q)}\} \leq \frac{1}{2}.$$

$$\min\{pq, (1-p)(1-q)\} \leq \frac{1}{4}.$$

## The full game of contingent plans

Only Player 2 is informed; first action in  $L$ , second action in  $R$ .

|     | $CC$      | $CD$               | $DC$               | $DD$          |
|-----|-----------|--------------------|--------------------|---------------|
| $C$ | $0, -2$   | $10p - 10, 5p - 7$ | $-10p, p - 2$      | $-10, 6p - 7$ |
| $D$ | $-1, -10$ | $4p - 5, p - 11$   | $-4p - 1, 5p - 10$ | $-5, 6p - 11$ |

$DC$  is strictly dominant when  $0 < p < 1$ .

At  $p = 0$  or  $p = 1$ , the plan at the zero-probability type is irrelevant.

## Exercise: who knows the quality of the car?

### Car values

|                                              | Seller $v_s$ | Buyer $v_b$ |
|----------------------------------------------|--------------|-------------|
| Good car                                     | 3            | 4           |
| Bad car                                      | 0            | 0           |
| Each quality has probability $\frac{1}{2}$ . |              |             |

### Trade at price $\pi$

Seller: keep or sell. Buyer: decline or buy.

Trade requires both to agree.

Net gains:  $(\pi - v_s, v_b - \pi)$ .

No trade: payoff 0 for each.

Compare: both informed / neither informed / only seller informed.

For which prices can trade occur in equilibrium?

## Why the maxmin never exceeds the minmax

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For any  $s_1, s_2$ ,

$$\min_{t_2} g(s_1, t_2) \leq g(s_1, s_2) \leq \max_{t_1} g(t_1, s_2).$$

Thus, for every  $s_2$ ,

$$\max_{s_1} \min_{t_2} g(s_1, t_2) \leq \max_{t_1} g(t_1, s_2).$$

$$\max_{s_1} \min_{s_2} g(s_1, s_2) \leq \min_{s_2} \max_{s_1} g(s_1, s_2).$$