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ORIGINAL PAPER

The complexity of interacting automata

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Abstract This paper studies the interaction of automata of size m. We characterise statistical properties satisfied by random plays generated by a correlated pair of automata with m states each. We show that in some respect the pair of automata can be identified with a more complex automaton of size comparable to $m \log m$. We investigate implications of these results on the correlated min–max value of repeated games played by automata.

Keywords Complexity · Automata · De Bruijn sequences · Bounded memory

1 Introduction

Automata are a central model in Game Theory when it comes to modelling agents with bounded cognitive abilities (see e.g. Aumann 1981; Abreu and Rubinstein 1988; Neyman 1997; Ben-Porath 1993). But how complex are they? Early on, Neyman (1998) and Kalai and Stanford (1988) noted that if a repeated game strategy σ includes

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exactly m continuation strategies, then m is also the size of the smallest automaton that can implement σ. This gives a foundation for automaton size as a measure for strategic complexity. In this paper, we take a different look at the same question by asking: how complex are phenomena that can be generated by one or several automata?

If we consider an isolated automaton of size m that generates a series of outputs in an alphabet A, then we know that this sequence is periodic of period at most m, and in fact, automata of size m can generate all such sequences. Since there are roughly speaking m^m such automata and $A^m \ll m^m$ such sequences, the automaton model is not economical when it comes to describing the behaviour of a single agent. The situation becomes more interesting when looking at interacting automata, which is the issue studied of this paper. Consider two agents interacting. Agent 1 is represented by an automaton σ^1 of size m and output space A^1 while agent 2 uses a repeated game strategy σ^2 with output space A^2. The pair (σ^1, σ^2) generates a sequence of outputs $a_1, a_2, \ldots, a_t$. An outside observer of the sequence who initially does not know (σ^1, σ^2), but knows that σ^1 is an automaton of size at most m, forms beliefs at every stage on the next action of σ^1. Since σ^1 is chosen in a finite set, the outside observer is, as t grows, eventually able to predict every action of agent 1. Actually, we show a stronger result. Theorem 2.1, called the information constraint, holds: the total per stage entropy of the outside observer's prediction on agent 1's action is bounded by the logarithm of the number of strategies of agent 1.

We prove several results showing that the information constraint is tight. Since the information constraint is obtained by looking solely at the number of automata of a given size, these converse results show that automata are as rich as any other model with the same number of strategies. In terms of complexity, automata generate phenomena of maximal randomness for the smallest possible number of strategies. In this sense, automata are a very economical model, i.e., do not have too many redundancies.

Our first converse result, Theorem 2.2, shows that, for any given distribution of beliefs $\mathcal{P}$ of an outside observer on the pair of actions of agents 1 and 2, there exist distributions over automata for agent 1 and strategies for agent 2 such that the average expected distribution of beliefs of an outside observer on both agents' actions between stages 1 and r is arbitrarily close to $\mathcal{P}$, as long as r is small enough so that the information constraint is satisfied. This result considers an outside observer who observes the actions of both agent 1 and 2, while the information constraint only considers agent 1's action. However, Theorem 2.2 particularized to distributions $\mathcal{P}$ in which the action of agent 2 is a function of the action of agent 1 shows that the information constraint provides a tight characterization of the beliefs an outside observer may have on the actions of an automaton that interacts with an outside strategy.

How about the complexity of two interacting automata, instead of one automaton interacting with an arbitrary strategy? Our second converse result shows that, when both agents are restricted to using automata of size at most m, the information constraint is tight up to a multiplicative constant that depends on the distribution of beliefs of the outside observer to be achieved. In particular, a pair of interacting automata of size m can generate a large subset of the set of sequences of period less than or equal to $m \log m$. We observe that $m \log m$ is much larger than the period m to which a single automaton is constrained. In terms of orders of magnitude, the logarithm of the number of pairs of automata is a constant times $m \log m$, and so is the logarithm of the number of sequences that can be generated. Therefore, this result shows that automata are not only a rich model when interacting with arbitrary strategies, but also when interactions are restricted within the model.

Therefore, it appears that while an automaton in isolation is unable to produce much complex phenomena, two interacting automata can. One may wonder what is the minimal complexity of a stream of inputs that an automaton requires in order to generate complex phenomena. Our Theorem 2.4 shows that an automaton of size m provided with a deterministic periodic stream of inputs whose period length is of order $\log m$ can generate random sequences of length of order $m \log m$.

The proof of our impossibility result relies on entropy techniques introduced by Lehrer (1988) and substantially developed by Neyman and Okada (1999, 2000, 2009) and then pursued by e.g. Gossner et al. (2006) and Peretz (2012, 2013). The converse results rely on extensions of the construction of automata whose states are elements of De Bruijn sequences (Gossner and Hernández 2003, 2006). In contrast to Gossner and Hernández (2003, 2006), we are not interested here in constructing an automaton that matches a given sequence, but in the construction of a random automaton or a pair of automata that achieve desired statistical properties. We rely on a result proven independently by Ornstein (1970) and Shapira (2007) showing that the desired property is satisfied when the entropy of the generated sequence is sufficiently close to the target.

The min–max values of two-player repeated games played by automata (Ben-Porath 1993; Neyman 1985, 1998; Kalai 1990; Neyman and Spencer 2010; Neyman 2008) or by strategies of bounded recall (Lehrer 1988; Peretz 2012) is relatively well understood. However, little is known about min–max values of games played with three or more finite automata, while a few results have been obtained on three players with bounded recall (see Bavly and Neyman 2014; Peretz 2013). Our results on the set of random plays generated by finite automata have natural consequences on the min–max values of repeated games played by three automata, that was one of the question which originally motivated our study.

The paper is organised as follows. Section 2 presents the model and the main results, which are then proven in Sect. 3. Section 4 examines the min–max values of repeated games played by finite automata.

2 Model and statement of the results

Let A^1 and A^2 be finite action sets for agents 1 and 2, and $A = A^1 \times A^2$. It is assumed throughout that $|A^1|, |A^2| \geq 2$. A (pure, reduced) strategy for agent $i \in \{1, 2\}$ in the repeated interaction is a function from $\cup_{t \geq 0}(A^{-i})^t$ to A^i, and the set of strategies for agent i is denoted Σ^i. A pair of strategies $(\sigma^1, \sigma^2) \in \Sigma^1 \times \Sigma^2$ induces a play $(a_1, \ldots, a_t, \ldots) \in A^{\mathbb{N}}$, where $a_t = (a_t^1, a_t^2)$ is defined recursively by $a_t^i = \sigma^i(a_1^{-i}, \ldots, a_{t-1}^{-i})$.

A (reduced) automaton of size m for agent i is given by a state space S of cardinality m, an initial state q_0, an action function $f: S \to A^i$, and a transition function $h: S \times A^{-i} \to S$. An automaton for agent i and a sequence $(a_1^{-i}, \ldots, a_t^{-i}) \in (A^{-i})^t$ induce a sequence of states $q_0, q_1, \ldots, q_t$ given recursively by

$$
\begin{aligned}
&q_0,\\
&q_1 = h(q_0, a_1^{-i}),\\
&\quad\vdots\\
&q_t = h(q_{t-1}, a_t^{-i}).
\end{aligned}
$$

Thus, an automaton defines a strategy σ^i by

$$\sigma^i(a_1^{-i}, \ldots, a_t^{-i}) = f(q_t).$$

We let $\Sigma^i(m)$ be the set of strategies of agent i induced by automata of size m.

For a compact metric space X, the set of probability measures on X endowed with the weak-* topology is a compact metric space which is denoted $\Delta(X)$. We abbreviate $\Delta(\Delta(X))$ by $\Delta\Delta(X)$. It is noted that throughout the paper we mainly consider finitely supported probability measures, so the weak-* topology does not play a crucial role.

Shannon's entropy of a probability measure over a finite space, $Q \in \Delta(X)$, is the quantity

$$H(Q) = -\sum_{x\in X} Q(x)\log\left(Q(x)\right),$$

where $\log = \log_2$ and $0\log 0 = 0$ by continuity. The entropy of a random variable $x \in X$, denoted $H(x)$, is the entropy of its distribution. If $x \in X$ and $y \in X$ are two finitely valued random variables $H(x, y)$ denotes the entropy of the random variable $(x, y) \in X \times Y$. The *entropy of x conditional on y* is defined by the chain rule $H(x|y) = H(x, y) - H(y)$. The *inequality of conditional entropy* asserts that $H(x) \geq H(x|y) \geq 0$, where the first inequality is equality if and only if x and y are independent. By taking π to be a uniform random permutation on X, the inequality of conditional entropy implies that $H(x) = H(\pi(x)|\pi) \leq H(\pi(x)) = \log|X|$, with equality if and only if x has the uniform distribution. The *mutual information* of x and y is defined as $I(x; y) = H(x) - H(x|y) = I(y; x)$. By the inequality of conditional entropy, $I(x; y) \geq 0$ with equality if and only if x and y are independent.

For $\mathcal{P} \in \Delta\Delta(X)$, we denote the expected entropy of $\mathcal{P}$ by

$$\bar{H}(\mathcal{P}) = \int H(Q)\,\mathrm{d}\mathcal{P}(Q).$$

A correlated strategy is a probability distribution $\tau \in \Delta(\Sigma^1 \times \Sigma^2)$. Such τ induces a probability distribution P_τ over the set of infinite histories $A^{\mathbb{N}}$. For any finite history $(a_1, \ldots a_{t-1})$ such that $P_\tau(a_1, \ldots, a_{t-1}) > 0$ we let

$$
\begin{aligned}
p_{t,\tau}(a_1, \ldots, a_{t-1}) &= P_\tau(a_t|a_1, \ldots, a_{t-1}),\\
p^i_{t,\tau}(a_1, \ldots, a_{t-1}) &= P_\tau(a^i_t|a_1, \ldots, a_{t-1}).
\end{aligned}
$$

That is, $p_{t,\tau}(a_1,\ldots,a_{t-1})$ and $p^i_{t,\tau}(a_1,\ldots,a_{t-1})$ represent the beliefs of an outside observer on a_t and on a^i_t given $a_1,\ldots,a_{t-1}$. We denote by $P_t(\tau)\in\Delta\Delta(A)$ and $P^i_t(\tau)\in\Delta\Delta(A^i)$ the laws of the random variables $p_{t,\tau}(a_1,\ldots,a_{t-1})$ and of $p^i_{t,\tau}(a_1,\ldots,a_{t-1})$ when $(a_1,\ldots,a_{t-1})$ is drawn according to P_τ.

Finally, for $r\geq 0$, we define the r-stage *expected empirical frequency of beliefs* (Gossner and Tomala 2006, 2007) induced by τ, $\mathcal{P}_r(\tau)\in\Delta\Delta(A)$ and $\mathcal{P}^i_r(\tau)\in\Delta\Delta(A^i)$, by

$$\mathcal{P}_r(\tau)=\frac{1}{r}\sum_{t=1}^{r}P_t,$$

$$\mathcal{P}^i_r(\tau)=\frac{1}{r}\sum_{t=1}^{r}P^i_t.$$

Note that we can express $\bar{H}(\mathcal{P}_r(\tau))=\frac{1}{r}\sum_{t=1}^{r}H(a_t|\bar{a}_{t-1})$ and $\bar{H}(\mathcal{P}^i_r(\tau))=\frac{1}{r}\sum_{t=1}^{r}H(a^i_t|\bar{a}_{t-1})$.

Clearly, the sets of all possible $\mathcal{P}_r(\tau)$ and $\mathcal{P}^i_r(\tau)$ when $r\geq 0$ and τ is an unrestricted correlated strategy are dense in $\Delta\Delta(A)$ and $\Delta\Delta(A^i)$ (since any $\mathcal{P}\in\Delta\Delta(A)$ can be approximated by an average of Dirac measures $\mathcal{P}_r=\frac{1}{r}(\delta_{Q_1}+\cdots\delta_{Q_r})$, and such $\mathcal{P}_r$ can be implemented by correlated strategies that play Q_i at each stage $i=1,\ldots,r$ independently of the history up to stage (i). Our aim is to investigate what restrictions are imposed on these sets when either or both strategies are taken among subclasses of bounded complexity.

By considering the cardinality of $\Sigma^i(m)$ we obtain the following constraint on the possible $\mathcal{P}^i_r(\tau)$ when σ^i is restricted to take values in $\Sigma^i(m)$.

Theorem 2.1 (Information constraint). *For every $m\in\mathbb{N}$, $\tau\in\Delta(\Sigma^1(m)\times\Sigma^2)$, and $r\in\mathbb{N}$,*

$$\frac{\bar{H}(\mathcal{P}^1_r(\tau))}{|A^2|-1}\leq\frac{m\log m+o(m\log m)}{r}.$$

Our main effort is dedicated to obtaining converse results showing that the bounds provided by the information constraint are tight. In order to state possibility results precisely, it will be convenient to consider a metric that induces the weak-* topology on $\Delta\Delta(A)$. The Wasserstein metric $\bar{d}$ serves this purpose. Let $\langle X,d\rangle$ be a metric space, e.g., $\langle\Delta(A),\|\cdot\|_1\rangle$. For $P,P'\in\Delta(X)$, the $\bar{d}$ distance between P and P' is defined by

$$\bar{d}(P,P')=\inf_{Q\in\Delta(X\times X),Q^1=P,Q^2=P'}\int d(x,y)\,\mathrm{d}Q(x,y),$$

where Q^1 and Q^2 denote the first and second marginals of Q respectively.

We are now in a position to state our first converse result to Theorem 2.1.

Theorem 2.2 *For every $\mathcal{P}\in\Delta\Delta(A)$ and $\epsilon>0$ there exists $r_0\in\mathbb{N}$ such that for every $r\geq r_0$ there exists $m\in\mathbb{N}$ and $\tau\in\Delta(\Sigma^1(m)\times\Sigma^2)$ such that*

$$\frac{\bar{H}(\mathcal{P})}{|A^2|-1}+\epsilon \geq \frac{m\log m}{r}$$

and

$$\bar{d}(\mathcal{P},\mathcal{P}_r(\tau))<\epsilon.$$

Theorem 2.2 shows that the information constraint of Theorem 2.1 is tight when particularized to distributions $\mathcal{P}$ such that $\bar{H}(\mathcal{P})=\bar{H}(\mathcal{P}^1)$, i.e., $H(a^2|a^1)=0$ ($\mathcal{P}$ a.s.). In terms of expected distributions of beliefs, the set of strategies $\Sigma^1(m)$ is as rich as any other set of strategies of the same cardinality can be. A natural question to ask is whether this richness still holds when the set in which σ^2 is chosen is also restricted.

In order to address this question we restrict both agents to strategies implementable through finite automata of size m. The information constraint shows that for every correlated $\tau\in\Delta(\Sigma^1(m)\times\Sigma_2(m))$,

$$\max_{i\in\{1,2\}}\left\{\frac{\bar{H}(\mathcal{P}_r^{-i}(\tau))}{(|A^i|-1)}\right\}\leq\frac{m\log m+o(m\log m)}{r}.$$

The next theorem asserts that the above bound is tight up to a constant multiplier that depends on $\mathcal{P}$. It shows that the information constraint is tight even when agent 2's strategy comes from the same (space) complexity class as that of agent 1.

Theorem 2.3 *For every $\mathcal{P}\in\Delta\Delta(A)$ there exists $C(\mathcal{P})>0$ such that for every $\epsilon>0$ there exists $r_0\in\mathbb{N}$ such that for every $r\geq r_0$ there exist $m\in\mathbb{N}$ and $\tau\in\Delta(\Sigma^1(m)\times\Sigma^2(m))$ such that*

$$C(\mathcal{P})+\epsilon\geq\frac{m\log m}{r}$$

and

$$\bar{d}(\mathcal{P},\mathcal{P}_r(\tau))<\epsilon.$$

Note that $C(\mathcal{P})$ does not depend on ϵ. It does, however, depend on $\mathcal{P}$. Whether one could replace $C(\mathcal{P})$ by a constant that does not depend on $\mathcal{P}$ (but rather just on A) is unknown. The information constraint implies that we cannot replace $C(\mathcal{P})$ by a constant $C<\max_{i\in\{1,2\}}\left\{\frac{\log(|A^{-i}|)}{|A^i|-1}\right\}$.

The next theorem provides a sense in which the information constraint is still tight under two further restrictions. First, the strategy of agent 2 is pure, i.e. constant on the support of τ. In particular, the strategies of 1 and 2 are now independent. Second, agent 2 uses a strategy in a much smaller space than $\Sigma^2(m)$, namely the space of periodic sequences[1] of period at most $\mathcal{O}(\log m)$. On the other hand, Theorem 2.4 considers

[1] Or, equivalently, in the space of oblivious automata of size at most $\mathcal{O}(\log m)$, i.e. those that ignore the actions of the other agent.

only the distribution of predictions of agent 1, and not of both 1 and 2's actions. In this result, the sequence of actions of agent 2 can be considered a "source" which, albeit deterministic, allows agent 1 to look more unpredictable from the point of view of an outside observer.

To fix notations, for $x > 0$ we denote by $A^2_p(x)$ the set of sequences of period at most x and identify each sequence $(a^2_t) \in A^2_p(x)$ with the strategy σ^2 given by $\sigma^2(a^1_1, \ldots, a^1_{t-1}) = a^2_t$ for every $(a^1_1, \ldots, a^1_{t-1})$.

Theorem 2.4 *For every $\mathcal{P}^1 \in \Delta\Delta(A^1)$ there exist $C = C(\mathcal{P}^1) > 0$ such that for every $\epsilon > 0$ there exists $r_0 \in \mathbb{N}$ such that for every $r \geq r_0$ there exist $m \in \mathbb{N}$ and $\tau \in \Delta(\Sigma^1(m)) \times A^2_p(C \log m)$ such that*

$$\frac{\bar{H}(\mathcal{P}^1)}{|A^2| - 1} + \epsilon \geq \frac{m \log m}{r}$$

and

$$\bar{d}(\mathcal{P}^1, \mathcal{P}^1_r(\tau)) < \epsilon.$$

3 Proofs of the main results

3.1 Information constraint

Theorem 2.1 is a consequence of the following upper bound on the number of finite automata.

Lemma 3.1

$$|\Sigma_i(m)| \leq m^{(|A^{-i}|-1)m+o(m)}.$$

Proof All elements of $\Sigma_i(m)$ are induced by automata with state space $\{1, \ldots, m\}$, thus are described by $q_0 \in \{1, \ldots, m\}$, $f : \{1, \ldots, m\} \to A^i$, and $h : \{1, \ldots, m\} \times A^{-i} \to \{1, \ldots, m\}$. This gives $m|A^i|^m m^{|A^{-i}|m}$ different descriptions. Since strategies are invariant to permutations of states in $\{1, \ldots, m\}$, by Stirling approximation, we have:

$$|\Sigma_i(m)| \leq |A^i|^m m^{|A^{-i}|m}/(m-1)! = m^{(|A^{-i}|-1)m+o(m)}.$$

□

To complete the proof of Theorem 2.1 we rely on a slight generalization of a result due to Neyman and Okada (1999) (Section 5) showing that the strategic entropy of a strategy is at most its entropy:

Lemma 3.2 *Let $\tau \in \Delta(\Sigma'^1 \times \Sigma^2)$, where Σ'^1 is any finite set and $r \geq 1$, then,*

$$\bar{H}(\mathcal{P}^1_r(\tau)) \leq \frac{1}{r} \log \left|\Sigma'^1\right|.$$

Proof Let $\tau \in \Delta(\Sigma'^1 \times \Sigma_2)$ be a correlated strategy. Denote by $\sigma = (\sigma^1, \sigma^2)$ a random variable with values in $\Sigma'^1 \times \Sigma^2$ with distribution τ and by $a_1, \ldots, a_r$, where $a_t = (a_t^1, a_t^2)$, the play induced by σ. We abbreviate $\bar{a}_t^i := a_1^i, \ldots, a_t^i$ $(i = 1, 2)$, and $\bar{a}_t := (\bar{a}_t^1, \bar{a}_t^2)$. By the chain rule of entropy, the inequality of conditional entropy, and the fact that $\bar{a}_t^1$ is a function of $\bar{a}_{t-1}^2$ and σ^1, we have:

$$\begin{aligned}\sum_{t=1}^{r} H(a_t^1|\bar{a}_{t-1}) + H(a_t^2|\bar{a}_{t-1}, a_t^1) &= H(\bar{a}_r) \\ &\leq H(\sigma^1, \bar{a}_r^2) = H(\sigma^1) + H(\bar{a}_r^2|\sigma^1) \\ &= H(\sigma^1) + \sum_{t=1}^{r} H(a_t^2|\bar{a}_{t-1}, a_t^1, \sigma^1).\end{aligned}$$

And therefore:

$$\bar{H}(\mathcal{P}_r^1(\tau)) = \frac{1}{r}\sum_{t=1}^{r} H(a_t^1|\bar{a}_{t-1}) \leq \frac{1}{r}H(\sigma^1) \leq \frac{1}{r}\log\left|\Sigma'^1\right|.$$

□

Theorem 2.1 is an immediate consequence of Lemmas 3.1 and 3.2.

3.2 Concatenable strategy pairs

The proofs of Theorems 2.2, 2.3 and 2.4 rely on the construction of adequate automata. These constructions are built in blocks, and larger automata are obtained by concatenation of smaller ones. We introduce a condition that ensures that such concatenations can be implemented by automata of size equal to the sum of the smaller ones.

We say that a pair given by an automaton σ^1 of agent 1 and a strategy σ^2 of agent 2 is *r-concatenable* if there exists a transition (s^1, a^2) such that, in the play induced by σ^1 and σ^2, the first hitting time of (s^1, a^2) is r. We denote by $(\Sigma^1(m^1) \times \Sigma^2)^r$ the subset of $\Sigma^1(m^1) \times \Sigma^2$ consisting of all r-concatenable pairs formed by an automaton of agent 1 of size m^1 and a strategy of agent 2.

A pair of automata (σ^1, σ^2) is *r-concatenable* if there exists a pair of transitions (s^1, a^2) for σ^1 and, (s^2, a^1) for σ^2 such that, in the play induced by σ^1 and σ^2, the first hitting time of both (s^1, a^2) and of (s^2, a^1) is r. We denote by $(\Sigma^1(m^1) \times \Sigma^2(m^2))^r$ the subset of $\Sigma^1(m^1) \times \Sigma^2(m^2)$ consisting of all r-concatenable pairs of automata σ^1 of size m^1 and σ^2 of size m^2.

A pair given by an automaton σ^1 of agent 1 and an r-periodic sequence of actions $\bar{a}^2 = a_1^2, \ldots, a_r^2, a_1^2 \ldots$ of agent 2 is *r-concatenable* if there exists a state s^1 of σ^1 such that, in the play induced by σ^1 and $\bar{a}^2$, the first hitting time of (s^1, a_r^2) is r. We denote by $(\Sigma^1(m^1) \times A_p^2(m^2))^r$ the subset of $\Sigma^1(m^1) \times A_p^2(m^2)$ consisting of all r-concatenable pairs formed by an automaton of agent 1 of size m^1 and a periodic sequence of actions of agent 2 whose period is at most m^2 (in particular, the period must divide r).

An r'-concatenable pair of an automaton and another strategy can be concatenated to an r''-concatenable such pair by redirecting the last transition of the former automaton to the initial state of the latter automaton. The result is an $r' + r''$-concatenable pair whose automaton size is the sum of the sizes of the two automata. This idea is formally expressed by the following three lemmata.

Lemma 3.3 (Concatenation Lemma 1). *Let $(\sigma_k^1, \sigma_k^2)_{1 \le k \le n}$ be a family of strategy pairs such that for every k, $(\sigma_k^1, \sigma_k^2) \in (\Sigma^1(m_k^1) \times \Sigma^2(m_k^2))^{r_k}$. Let $r = \sum_k r_k$, $m^1 = \sum_k m_k^1$, and $m^2 = \sum_k m_k^2$. Then, there exists a strategy pair $(\sigma^1, \sigma^2) \in (\Sigma^1(m^1) \times \Sigma^2(m^2))^r$ such that the play induced by (σ^1, σ^2) is the concatenation of the length r_k plays induced by $(\sigma_k^1, \sigma_k^2)_{1 \le k \le n}$ respectively.*

Lemma 3.4 (Concatenation Lemma 2). *Let $(\sigma_k^1, \sigma_k^2)_{1 \le k \le n}$ be a family of strategy pairs such that for every k, $(\sigma_k^1, \sigma_k^2) \in (\Sigma^1(m_k^1) \times \Sigma^2)^{r_k}$. Let $r = \sum_k r_k$, and $m^1 = \sum_k m_k^1$. Then, there exists a strategy pair $(\sigma^1, \sigma^2) \in (\Sigma^1(m^1) \times \Sigma^2)^r$ such that the play induced by (σ^1, σ^2) is the concatenation of the length r_k plays induced by $(\sigma_k^1, \sigma_k^2)_{1 \le k \le n}$ respectively.*

Lemma 3.5 (Concatenation Lemma 3). *Let $(\sigma_k^1)_{1 \le k \le n}$ be a family of strategies for agent 1 and $\bar{a}^2$ an m^2-periodic sequence for a agent 2. Suppose that for every k, $(\sigma_k^1, \bar{a}^2) \in (\Sigma^1(m_k^1) \times A_p^2(m^2))^{r_k}$. Let $r = \sum_k r_k$, $m^1 = \sum_k m_k^1$. Then, there exists a strategy σ^1 for agent 1 such that $(\sigma^1, \bar{a}^2) \in (\Sigma^1(m^1) \times A_p^2(m^2))^r$ and the play induced by $(\sigma^1, \bar{a}^2)$ is the concatenation of the length r_k plays induced by $(\sigma_k^1, \bar{a}^2)_{1 \le k \le n}$ respectively.*

The next lemma, Lemma 3.6, utilises the concatenation lemmata in order to reduce Theorems 2.2, 2.3, and 2.4, to considering only r's in a sequence that does not increase too fast. Formally, Lemma 3.6 consists of three lemmata, referring to the three theorems. Since the three lemmata are very similar both in their formulation and in their proofs, we state and prove them as one result.

Lemma 3.6 *Let $\{m_k\}$, $\{r_k\}_{k=1}^{\infty}$ be sequences of natural numbers, and let $\tau_k \in \Delta(\Sigma^1(m_k) \times \Sigma^2)^{r_k}$ (respectively, $\Delta(\Sigma^1(m_k) \times \Sigma^2(m_k))^{r_k}$, or $\Delta(\Sigma^1(m_k) \times A_p^2(f(m_k)))^{r_k}$ for some non-decreasing $f : \mathbb{N} \to \mathbb{N}$ and with pure marginal τ_k^2).*
If

(i) $\sup_{k \in \mathbb{N}} r_k = \infty$, *and*
(ii) $\sup_{k \in \mathbb{N}} \frac{r_{k+1}}{r_k} < \infty$,

then, for every $\mathcal{P} \in \Delta\Delta(A)$ and every $\epsilon > 0$ there exists $r_0 \in \mathbb{N}$ such that for every $r > r_0$ there are $r' \le r$, $m \in \mathbb{N}$, and $\tau \in \Delta(\Sigma^1(m), \Sigma^2)^{r'}$ (respectively, $\Delta(\Sigma^1(m) \times \Sigma^2(m))^{r'}$, or $\Delta(\Sigma^1(m) \times A_p^2(f(m)))^{r'}$ with τ^2 pure) satisfying

(a) *τ is the concatenation of $\left\lfloor \frac{r}{r_k} \right\rfloor$ independent copies of τ_k, for some $k \in \mathbb{N}$,*
(b) $r' = \left\lfloor \frac{r}{r_k} \right\rfloor \cdot r_k > (1 - \epsilon) r$,
(c) $m = \left\lfloor \frac{r}{r_k} \right\rfloor \cdot m_k$,

(d) $\frac{m \log m}{r'} < \limsup_{k\to\infty} \frac{m_k \log m_k}{r_k} + \epsilon$,
(e) $\bar{d}(\mathcal{P}, \mathcal{P}_{r'}(\tau)) < \limsup_{k\to\infty} \bar{d}(\mathcal{P}, \mathcal{P}_{r_k}(\tau_k)) + \epsilon$.

Proof Assume w.l.o.g. that $\{r_k\}$ is increasing (otherwise consider an increasing subsequence). Let $\mathcal{P} \in \Delta\Delta(A)$, $1 > \epsilon > 0$, and $k_0 \in \mathbb{N}$ (later, k_0 will be assumed to be sufficiently large). Let $r_0 = \lceil \epsilon^{-1} r_{k_0} \rceil$. For $r \geq r_0$, let k be the largest integer such that $r_k \leq \epsilon r$. Let $C = \sup_{k\in\mathbb{N}} \frac{r_{k+1}}{r_k}$. Let $r' = r_k \left\lfloor \frac{r}{r_k} \right\rfloor$. It follows that

$$r' > (1-\epsilon)r.$$

Let τ_k be the strategy assumed by the lemma. Set τ to be the concatenation of $\left\lfloor \frac{r}{r_k} \right\rfloor$ independent copies of τ_k. It follows that

$$\mathcal{P}_{r'}(\tau) = \mathcal{P}_{r_k}(\tau_k),$$

and so, choosing k_0 large enough ensures the requirement

$$\bar{d}(\mathcal{P}, \mathcal{P}_{r'}(\tau)) < \limsup_{k\to\infty} \bar{d}(\mathcal{P}, \mathcal{P}_{r_k}(\tau_k)) + \epsilon.$$

Let $m = \left\lfloor \frac{r}{r_k} \right\rfloor m_k$. The concatenation lemma ensures that $\tau \in \Delta(\Sigma^1(m) \times \Sigma^2)^{r'}$. Respectively, $\tau \in \Delta(\Sigma^1(m) \times \Sigma^2(m))^{r'}$, or $\Delta(\Sigma^1(m) \times A_p^2(f(m)))^{r'}$ with τ^2 pure. The latter holds because $m \geq m_k$, f is non-decreasing, and the definition of $A_p(x)$ requires that the period is at most x.

It remains to verify that $\frac{m \log m}{r'} < \limsup_{k\to\infty} \frac{m_k \log m_k}{r_k} + \epsilon$, for an appropriate choice of k_0. Since $Cr_k \geq r_{k+1} > \epsilon r \geq \epsilon r'$, we have $\left\lfloor \frac{r}{r_k} \right\rfloor = \frac{r'}{r_k} < \epsilon^{-1} C$. Therefore,

$$\frac{m \log m}{r'} < \frac{m_k \log m_k}{r_k} + \frac{m_k}{r_k} \log(\epsilon^{-1} C).$$

Assuming $\limsup_{k\to\infty} \frac{m_k \log m_k}{r_k} < \infty$, for any k large enough, we have $\frac{m_k \log m_k}{r_k} < \limsup_{k\to\infty} \frac{m_k \log m_k}{r_k} + \frac{\epsilon}{2}$ and $\frac{m_k}{r_k} \log(\epsilon^{-1} C) < \frac{\epsilon}{2}$. □

3.3 Building blocks

We define a few building blocks that will be used in the construction of the strategies required for Theorems 2.2, 2.3, and 2.4 . For $l \in \mathbb{N}$ and $x \in (A)^l$, the *empirical frequency*, $\mathrm{emp}(x) \in \Delta(A)$, is defined by

$$\mathrm{emp}(x)(a) = \frac{1}{l} \left|\{t \in [l] : x_t = a\}\right|,$$

where $[n] := \{1, \ldots, n\}$.

For a rational distribution Q over A, we let $T^Q(l)$ be the set of sequences of elements of A of length l with empirical frequency Q. We say that l is a common denominator of Q if $T^Q(l) \neq \emptyset$. We use the fact that if l_0 is a common denominator of Q, then (Cover and Thomas 2006, Chapter 11)

$$\begin{aligned} \frac{\log |T^Q(l_0)|}{l_0} &\leq H(Q), \text{ and} \\ \lim_{u\to\infty} \frac{\log |T^Q(ul_0)|}{ul_0} &= H(Q). \end{aligned} \tag{3.1}$$

Throughout, an *alphabet* is a set that contains at least two distinct elements. A *De Bruijn sequence of order k over a finite alphabet B* is a $|B|^k$-periodic sequence of B symbols $x_1, x_2, \ldots$, such that for every $(b_1, \ldots, b_k) \in B^k$ there exists a unique $1 \leq t \leq |B|^k$ such that $(b_1, \ldots, b_k) = (x_t, \ldots, x_{t+k-1})$. The existence of De Bruijn sequences was shown, e.g., by De Bruijn (1946) through the existence of Eulerian cycles in De Bruijn's graph.

The empirical frequency of De Bruijn sequences is always uniform. In order to obtain sequences with similar properties, but whose empirical frequency is arbitrary we define De Bruijn sequences over compound alphabets. For $l \geq 1$, let Y be a subset of B^l with at least two distinct element, called the set of words. A *compound De Bruijn sequence of order k over Y* is an $l|Y|^k$-periodic sequence of B elements obtained by the concatenation of the $|Y|^k$ words forming a De Bruijn sequence of order k over the alphabet Y. It is, therefore, an $l|Y|^k$-periodic sequence $x_1, x_2, \ldots$ of elements in B such that for every sequence $(y_1, \ldots, y_k) \in Y^k$, there exists a unique $1 \leq j \leq |Y|^k$ such that $(y_1 \cdots y_k) = (x_{jl+1}, \ldots, x_{jl+kl})$ (where uv denotes the concatenation of u and v). The item $x_t \in B$ is the *tth element*, while we call $y_j = (x_{jl+1}, \ldots, x_{jl+l}) \in B^l$ the *jth word*, and $(y_j, \ldots, y_{j+k-1}) \in B^{kl}$ the *jth block*. When $Y = T^Q(l)$, the compound De Bruijn has empirical frequency Q.

The following definition provides a simple upper bound on the amount of information needed to describe a finite string of symbols given another string. We define a variant of Levenshtein's edit distance. For a set X we let $X^{<\infty}$ be the set of finite sequences of elements of X. Given a finite alphabet B, the *simple edit operations* on $B^{<\infty}$ are the following $2|B| + 2$ operations: appending one symbol at the end or the beginning of a string, and deleting the first or last element of a string. We define the *simple edit distance* between two words $w, u \in B^{<\infty}$, denoted $e(w, u)$, to be the minimal number of simple edit operations needed in order to transform w to u. As usual, the distance between a string w and a set of strings K, $e(w, K)$, is the minimal distance between w and some $u \in K$. Note that, up to a multiplier of $\log(2|B| + 2)$, the simple edit distance is an upper bound on the amount of information content of one string given another string.

The strategies required for Theorems 2.2, 2.3, and 2.4 all have a common structure. We next describe the common structure of the random plays induced by these strategies.

Definition 3.7 Let $Y \subset A^l$, for some $l \geq 1$. A *random enumeration scheme over Y* is a sequence of tuples $\langle L_k, G_k\rangle_{k=1}^{\infty}$, where for every k, $L_k = (w_1, \ldots, w_{|L_k|})$ is a finite sequence of strings in $A^{<\infty}$ and G_k is a group of permutation on $[|L_k|]$. In addition, there is a constant C (that does not depend on k) such that

(i) $|\{s \in [|L_k|] : w_s = w_t\}| \le C \quad \forall t \in [|L_k|]$,
(ii) $e(w_t, Y^k) \le C \quad \forall t \in [|L_k|]$,
(iii) $|Y|^k/|L_k| \le C$,
(iv) $\left(\frac{|L_k|!}{|G_k|}\right)^{\frac{1}{|L_k|}} \le C$,

for all $k \in \mathbb{N}$.

Each element of the group G_k acts on the sequence L_k by re-ordering its elements. That is, every $\pi \in G_k$ transforms L_k into

$$(w_{\pi(1)}, \ldots, w_{\pi(|L_k|)}).$$

When $\pi \in G_k$ is random and uniformly distributed, the corresponding random sequence $\bar{a}(k) := w_{\pi(1)} \cdots w_{\pi(|L_k|)}$ is called the *induced random play*.

Informally, the idea is that L_k is approximately an enumeration of Y^k and G_k contains a substantial portion of all the permutations of L_k, so that the induced random play $\bar{a}(k)$ approximates a sequence of i.i.d. random variables drawn uniformly from Y. The two processes are similar in two ways: they produce similar sequences and they have similar entropy rates. Since for each element w_t of L_k, $e(w_t, Y^k) \le C$, w_t is almost entirely composed of a concatenation of strings from Y, and hence so is $\bar{a}(k)$. The next lemma says that the entropy rate of $\bar{a}(k)$ is also similar to that of a uniformly distributed independent sequence of Y-valued random variables.

Lemma 3.8 *Let $\langle L_k, G_k\rangle_{k=1}^{\infty}$ be a random enumeration scheme over a compound alphabet $Y \subset A^l$. Let r_k be the total length of the random play $\bar{a}(k)$. Then,*

$$\liminf_{k\to\infty} \frac{1}{r_k} H(\bar{a}(k)) \ge \frac{\log|Y|}{l}.$$

Proof Let π be a uniformly random permutation from G_k, and $\bar{a}(k) = a_1, \ldots, a_{r_k}$ the corresponding induced play. We first estimate $\frac{1}{r_k}H(\pi)$ and then compare it to $\frac{1}{r_k}H(\bar{a}(k))$.

Let C be as in Definition 3.7. Since the length of any word in L_k is at least $kl - C$ and at most $kl + C$, we have

$$|L_k|(kl - C) \le r_k \le |L_k|(kl + C).$$

Since $|G_k| \ge C^{-|L_k|}|L_k|!$ and $|L_k| \ge C^{-1}|Y|^k$,

$$\begin{aligned}\frac{1}{r_k}H(\pi) &= \frac{1}{r_k}\log(|G_k|) \ge \frac{\log(|L_k|!)}{r_k} - \frac{|L_k|\log C}{r_k} \ge \frac{\log(|L_k|!)}{|L_k|(kl+C)} - \frac{\log C}{kl+C}\\ &\ge \frac{\log(|L_k|)(1-o(1))}{kl+C} \ge \frac{k\log(|Y|)(1-o(1))}{kl+C} \ge \frac{\log|Y|}{l}(1-o(1)).\end{aligned}$$

We now compare $\frac{1}{r_k}H(\bar{a}(k))$ to $\frac{1}{r_k}H(\pi)$ by coupling $\bar{a}(k)$ with additional random variables of low entropy, such that π can be read off from $\bar{a}(k)$ and the additional random variables. Recall that

$$\bar{a}(k) = w_{\pi(1)} w_{\pi(2)} \cdots w_{\pi(|L_k|)},$$

where $L_k = (w_1, w_2, \ldots, w_{|L_k|})$. Denote by $|w|$ the length of the word w. For each $j \in [|L_k|]$, let

$$\begin{aligned} b_j &= |w_{\pi(j)}|, \text{ and} \\ c_j &= \left|\left\{i < \pi(j) : w_i = w_{\pi(j)}\right\}\right|. \end{aligned}$$

Note that from $\bar{a}$ and $b_1, \ldots, b_{|L_k|}$ one can read off the words $w_{\pi(1)}, \ldots, w_{\pi(|L_k|)}$, and further from $c_1, \ldots, c_{|L_k|}$ the permutation π itself. Notice that the sequence $w_1, \ldots, w_{|L_k|}$ is not random, and therefore is treated as given information. The values of each b_j range between $kl - C$ and $kl + C$, and the values of each c_j are in $[C]$. It follows that

$$\begin{aligned} \frac{1}{r_k}H(\bar{a}(k)) &\geq \frac{1}{r_k}H(\pi) - \frac{1}{r_k}[H(b_1, \ldots, b_{|L_k|}) + H(c_1, \ldots, c_{|L_k|})] \\ &\geq \frac{\log|Y|}{l}(1 - o(1)) - \frac{|L_k|}{r_k}\log((2C+1)C) \geq \frac{\log|Y|}{l}(1 - o(1)). \end{aligned}$$

□

We now describe a framework within which one can construct schemes of automata that are naturally associated with certain random enumeration schemes.

Definition 3.9 A *De Bruijn automaton scheme for agent i* is a tuple $\Xi = \left\langle \mathcal{P}, l, \left\{x(k), L_k, (s_t(k), z_t(k), a_t^{-i}(k))_{t=1}^{|L_k|}, G_k\right\}_{k=1}^{\infty}\right\rangle$ where

- $\mathcal{P} = \sum_j q_j \delta_{Q_j} \in \Delta\Delta(A)$ is a finitely supported rational distribution over rational beliefs.
- $l \in \mathbb{N}$ is such that each $l_j := q_j l$ is a common denominator of Q_j (in particular, l_j is an integer).

Let $Y = \times_j T^{Q_j}(l_j)$. For every $k \in \mathbb{N}$,

- $x = x(k) = (x_1, \ldots, x_{m_k})$ is a compound De Bruijn sequence of order k over Y, where $m_k = l|Y|^k$.
- $L_k = (w_1, \ldots, w_{|L_k|})$ is a sequence of A-words such that $\sup_{k\in\mathbb{N}} m_k/|L_k| < \infty$.
- For $t = 1, \ldots, |L_k|$,
  - $s_t = s_t(k) \in [m_k]$,
  - $z_t = z_t(k) \in [lk] \setminus [l(k-1)]$, and
  - $a_t^{-i} = a_t^{-i}(k) \in A^{-i} \setminus \left\{x_{s_t}^{-i}\right\}$,

such that

$$w_t = x_{s_t - z_t} \cdots x_{s_t - 1} \, (x^i_{s_t}, a^{-i}_t),$$

and $(s_t, a^{-i}_t) \neq (s_{t'}, a^{-i}_{t'})$ for all $t \neq t'$.

- G_k is a group of permutations on $[|L_k|]$, with $\sup_{k \in \mathbb{N}} \left(\frac{|L_k|!}{|G_k|} \right)^{\frac{1}{|L_k|}} < \infty$.

Definition 3.10 Given a De Bruijn automaton scheme Ξ, an automaton $\sigma^i_\Xi(k) \in \Sigma^i(m_k)$ is defined by[2]:

- the state space $[m_k]$;
- the initial state $s_1 - z_1 \bmod m_k$;
- the action function $f(s) = x^i_s$;
- the transition function

$$h(s, a^{-i}) = \begin{cases} s + 1 \bmod m_k & \text{if } a^{-i} = x^{-i}_s, \\ s_{t+1} - z_{t+1} \bmod m_k & \text{if } s = s_t \text{ and } a^{-i} = a^{-i}_t \text{ for some } t \in [|L_k|], \\ \text{unspecified} & \text{otherwise.} \end{cases}$$

A transition of the first type is called a *+1 transition* and a transition of the second type a *jump transition*.

Note that the jump transitions are well defined, since it is assumed that the pairs (s_t, a^{-i}_t) are distinct.

The rest of Sect. 3.3 presents relevant properties of random enumeration schemes. For readability, the De Bruijn automaton schemes are assumed to be for agent 1. Of course, similar properties hold for random enumeration schemes for agent 2.

With the notation of Definition 3.9, the concatenation of the elements of L_k is denoted

$$\bar{a}_\Xi(k) = w_1 w_2 \cdots w_{|L_k|}.$$

The next lemma ensures that the strategy $\sigma^1_\Xi(k)$ generates the desired play $\bar{a}_\Xi(k)$, has the necessary concatenability properties and provides the relevant upper bound on the state space cardinality.

Lemma 3.11 *Let* $\Xi = \left\langle \mathcal{P}, l, \left\{ x(k), L_k, (s_t(k), z_t(k), a^2_t(k))_{t=1}^{|L_k|}, G_k \right\}_{k=1}^{\infty} \right\rangle$ *be a De Bruijn automaton scheme for agent 1, then:*

(i) *the induced strategy* $\sigma^1_\Xi(k)$ *is consistent with the play* $\bar{a}_\Xi(k)$,
(ii) *the induced strategies* $\sigma^1_\Xi(k)$ *coupled with (any strategy consistent with)* $\bar{a}_\Xi(k)$ *is* r_k*-concatenable, and*
(iii) $\limsup_{k\to\infty} \frac{m_k \log m_k}{r_k} \le \bar{H}(\mathcal{P}) \limsup_{k\to\infty} \frac{m_k}{|L_k|}$.

[2] Here and throughout, $s \bmod m$ is defined as the number in $[m]$ which is equal to s modulo m.

Proof Part (i) follows from the definition of $\sigma^1_\Xi(k)$. In the beginning and after each jump transition, $\sigma^1_\Xi(k)$ is in state $s_t - z_t$. In the next z_{t+1} periods $\bar{a}_\Xi(k)$ prescribes the action profiles $(x_{s_t-z_t+j-1} : j \in [z_t])$. Provided $\sigma^1_\Xi(k)$ is at state $s_t - z_t + j - 1$ it plays $x^1_{s_t-z_t+j-1}$, as required. Provided agent 2, too, plays as required, $x^2_{s_t-z_t+j-1}$, a "+1" transition to $s_t - z_t + j$ occurs. After z_t steps, $\sigma^1_\Xi(k)$ is at state s_t. At that time $\bar{a}_\Xi(k)$ prescribes the action profile $(x^1_{s_t}, a^2_t)$. If agent 2 plays a^2_t, then a jump transition to $s_{t+1} - z_t$ occurs, and the result follows by induction on t.

Part (ii) holds since the play ends at a jump transition and each jump transition occurs at most once by Definition 3.9.

For Part (iii) note that $r_k = \sum_{t=1}^{|L_k|} z_t + 1 \geq |L_k|(k-1)l$. Thus,

$$\frac{m_k \log m_k}{r_k} \leq \frac{\log m_k}{kl} \frac{k}{k-1} \frac{m_k}{|L_k|}.$$

It remains to show that $\limsup \frac{\log m_k}{kl} \leq \bar{H}(\mathcal{P})$. Recall that $m_k = l|Y|^k$. Since $\log |Y| = \sum_i \log |T^{Q_i}(l_i)| \leq \sum_i l_i H(Q_i) = l\bar{H}(\mathcal{P})$,

$$\frac{\log m_k}{kl} = \frac{\log |Y|}{l} + \frac{\log l}{kl} \leq \bar{H}(\mathcal{P}) + \frac{\log l}{kl} \xrightarrow[k\to\infty]{} \bar{H}(\mathcal{P}).$$

□

A permutation $\pi \in G_k$ acts on the kth component of Ξ by transforming $L_k = (w_1, \ldots, w_{|L_k|})$ into $(w_{\pi(1)}, \ldots, w_{\pi(|L_k|)})$, s_t into $s_{\pi(t)}$, z_t into $z_{\pi(t)}$, and a^2_t into $a^2_{\pi(t)}$ for $t = 1, \ldots, |L_k|$. This transformation defines another De Bruijn automaton scheme denoted $\pi.\Xi$. The following lemma describes the relation between De Bruijn automaton schemes and random enumeration schemes.

Lemma 3.12 *Let*

$$\Xi = \left\langle \mathcal{P} = \sum_j \frac{l_j}{l} \delta_{Q_j}, l, \left\{ x(k), L_k, (s_t(k), z_t(k), a^2_t(k))_{t=1}^{|L_k|}, G_k \right\}_{k=1}^{\infty} \right\rangle$$

be a De Bruijn automaton scheme for agent 1. Then:

(i) $\langle L_k, G_k \rangle_{k=1}^{\infty}$ *is a random enumeration scheme over* $Y = \times_j T^{Q_j}(l_j)$*, and*
(ii) $\sigma^1_{\pi.\Xi}$ *is consistent with the play* $w_{\pi(1)} w_{\pi(2)} \cdots w_{\pi(|L_k|)}$ *(where* $L_k = (w_1, w_2, \ldots, w_{|L_k|})$*).*

Proof *Part* (ii). By the definition of $\pi.\Xi$,

$$\bar{a}_{\pi.\Xi}(k) = w_{\pi(1)} w_{\pi(2)} \cdots w_{\pi(|L_k|)}.$$

By Lemma 3.11, $\sigma^1_{\pi.\Xi}$ is consistent with $\bar{a}_{\pi.\Xi}(k)$.

Part (i). We need to verify (i)–(iv) from Definition 3.7. (iii) and (iv) follow from Definition 3.9, since $m_k = l|Y|^k$ and $\sup_{k\in\mathbb{N}} m_k/|L_k| < \infty$.

We prove (ii). Let $L_k = (w_1, \ldots, w_{|L_k|})$. Each word w_t consists of z_t consecutive elements $x_{s_t - z_t}, \ldots, x_{s_t - 1}$ of x followed by a single symbol. Since $|z_t - kl| \leq l$, there is some block $B = [(u+k)l] \setminus [ul]$ such that the symmetric difference between B and $\{s_t - z_t, \ldots, s_t - 1\}$ is at most $2l$; therefore the simple edit distance between w_t and x_B is at most $2l + 1$. (ii) then holds since $x_B \in Y^k$.

We finally show (i). We must bound from above the multiplicity of elements in L_k. Fix a mapping $t \mapsto B(t)$ as described above, i.e., $B(t) = [(u+k)l] \setminus [ul]$ such that $s_t \in [(u+k)l] \setminus [(u+k-1)l]$. We must bound from above

$$\max_{t_0 \in [|L_k|]} |\left\{t : w_t = w_{t_0}\right\}|,$$

which is clearly not more than

$$\left(\max_B |\{t : B(t) = B\}|\right) \left(\max_{t_0 \in [|L_k|]} |\left\{B(t) : w_t = w_{t_0}\right\}|\right).$$

We bound each one of the above factors from above by functions of l and $|A|$ that do not depend on k. We begin with the first factor, the number of indices t mapped to any given block B. Since $s_t \in [(u+k)l] \setminus [(u+k-1)l]$, there are only l possible values for s_t. Since each (s_t, a_t^2) is unique, the multiplicity of each s_t is at most $|A^2| - 1$; therefore the number of indices mapped to any given B is at most $l(|A^2| - 1)$.

Now, fix $t_0 \in [|L_k|]$. We bound from above the cardinality of $\left\{B(t) : w_t = w_{t_0}\right\}$. Since the simple edit distance between w_t and $x_{B(t)}$ is at most $2l + 1$, for any t, we have that if $w_t = w_{t_0}$ then $x_{B(t)}$ belongs to the ball of radius $4l + 2$ around $x_{B(t_0)}$. Since the size of that ball is a function of l and $|A|$ (but not k) and since x_B determines B, the cardinality of $\left\{B(t) : w_t = w_{t_0}\right\}$ is bounded (by a function of l and $|A|$). □

A De Bruijn automaton scheme allows us to implement a random play with certain properties. Lemma 3.13 shows that these properties guarantee that the expected distribution of beliefs induced by the play is close to the target distribution $\mathcal{P}$.

Lemma 3.13 *Let Ξ be a De Bruijn automaton scheme for agent 1 and $\mathcal{P} = \sum_{i=1}^n q_i \delta_{Q_i}$. Let r_k be the length of $\bar{a}_\Xi(k)$. For every $k \in \mathbb{N}$, let $\bar{a}(k) = \bar{a}_{\pi.\Xi}(k)$ be the random play obtained by taking a uniform random permutation $\pi \in G_k$. The corresponding mixture of strategies $\sigma^1_{\pi.\Xi}(k)$ coupled with any $\bar{a}_{\pi.\Xi}(k)$ consistent strategies for agent 2 defines a correlated strategy τ_k.*

(i) *The play induced by τ_k is $\bar{a}(k)$.*
(ii) *The strategy τ_k is in $\Delta(\Sigma^1(m_k) \times \Sigma^2)^{r_k}$.*
(iii) *Let $Q \in \Delta(A \times [n])$ be given by $Q(a, i) = q_i Q_i(a)$. The random play $\bar{a} = \bar{a}(k)$ can be coupled with a random $[n]$-valued sequence $\bar{b} = b_1, \ldots, b_{r_k}$, such that*
- $\lim_{k\to\infty} \left\| Q - \mathbb{E}\left[\mathrm{emp}(\bar{a}, \bar{b})\right]\right\| = 0,$
- $\lim_{k\to\infty} \frac{1}{r_k} H(\bar{b}) = 0.$

(iv) $\liminf_{k\to\infty} \frac{1}{r_k} H(\bar{a}(k)) \geq \bar{H}(\mathcal{P}) - f(l)$*, where $f(l) \to 0$, as $l \to \infty$, and f depends only on $\mathcal{P}$.*
(v) $\limsup_{k\to\infty} \frac{m_k \log m_k}{r_k} \leq \bar{H}(\mathcal{P}) \limsup_{k\to\infty} \frac{m_k}{|L_k|}.$

Proof Parts (i), (ii), and (v) immediately follow from Lemmas 3.11 and 3.12.

We prove Part (iv). Lemma 3.12 (i) says that $\langle L_k, G_k\rangle_{k=1}^{\infty}$ is a random enumeration scheme over Y; therefore, by Lemma 3.8, $\liminf_{k\to\infty} \frac{1}{r_k} H(\bar{a}(k)) \geq \frac{\log|Y|}{l}$. By Eq. (3.1), $\lim_{l\to\infty} \frac{\log|Y|}{l} = \bar{H}(\mathcal{P})$.

It remains to verify Part (iii). Associate with any state of $s \in [m_k]$ a number $i = i(s) \in [n]$, defined by

$$s \in [l_1 + \cdots + l_i] \setminus [l_1 + \cdots + l_{i-1}] \quad \bmod l.$$

Recall that the induced play is of the form $\bar{a} = w_1 w_2 \cdots w_{|L_k|}$, where $w_t = (x_{s_t^1 - z_t}, \ldots, x_{s_t - 1}, a_t)$, for some $a_t \in A$. Let $b_t = i(s_t - z_t) \cdots i(s_t - 1)\, 1$, and $\bar{b} = (b_1, b_2, \ldots, b_{|L_k|})$. Since $|w_t| = |b_t| = z_t + 1 > (k-1)l$, and since the empirical distribution of any word $x_{tl+1} \ldots x_{tl+l}$ coupled with $i(tl+1) \ldots i(tl+l)$ is Q, $\|\mathrm{emp}(w_t, b_t) - Q\| = \mathcal{O}(k^{-1})$; therefore $\left\| Q - \mathbb{E}[\mathrm{emp}(\bar{a}, \bar{b})]\right\| = \mathcal{O}(k^{-1})$.

The sequence $\bar{b}$ can be read off from the random variables $\{z_t\}$ and $\{s_t \bmod l\}$, $t = 1, \ldots, |L_k|$. Since these random variables take values in sets of size l,

$$\frac{1}{r_k} H(\bar{b}) \leq \frac{1}{r_k} H(z_t, s_t \bmod l : 1 \leq t \leq |L_k|) \leq \frac{2|L_k| \log l}{r_k} \leq \frac{2 \log l}{(k-1)l} \to 0,$$

as $k \to \infty$. □

3.4 Information criterion

The following lemma will allow us to determine whether a distribution of beliefs is $\bar{d}$-close to a target Dirac distribution.

Lemma 3.14 *Let A be a finite set and $H\colon \Delta(A) \to \mathbb{R}$ continuous and strictly concave. For every $\epsilon > 0$ there is $\delta > 0$ such that for every $Q \in \Delta(A)$ and $\mathcal{P} = \sum_i q_i \delta_{Q_i} \in \Delta\Delta(A)$, if*

(i) $\left\| Q - \sum_i q_i Q_i \right\| < \delta$, *and*
(ii) $\sum_i q_i H(Q_i) > H(Q) - \delta$,

then

$$\bar{d}(\delta_Q, \mathcal{P}) < \epsilon.$$

Lemma 3.14 has been shown by several authors independently, see for instance Lemma 1 in Ornstein (1970) and Lemma 22 in Shapira (2007).

A distribution over beliefs can be represented by a pair of random variables. Let x be a random variable taking values in a finite set X. Define $\mathrm{p}(x) \in \Delta\Delta(X)$ to be the Dirac measure supported on the distribution of x. For an event of positive probability E, define $\mathrm{p}(x|E) \in \Delta\Delta(X)$ to be the Dirac measure supported on the distribution of x conditional on E. Let y be another random variable taking values in a finite set Y.

The distribution over beliefs $\mathrm{p}(x|y) \in \Delta\Delta(X)$ is defined by

$$\sum_{\mathbf{y} \in Y} \mathbf{P}(y = \mathbf{y})\mathrm{p}(x|y = \mathbf{y}).$$

Lemma 3.14 implies that for every $\epsilon > 0$ there is $\delta = \delta(\epsilon, |X|)$, such that

$$I(x; y) < \delta \quad \Rightarrow \quad \bar{d}(\mathrm{p}(x), \mathrm{p}(x|y)) < \epsilon.$$

Let z be a third random variable. By Markov's inequality we have

$$I(x; y|z) < \delta^2 \quad \Rightarrow \quad \bar{d}(\mathrm{p}(x|z), \mathrm{p}(x|y, z)) < \epsilon + \delta. \tag{3.2}$$

Explanation: Assume the left hand side of (3.2) holds. Let $Z = \{\zeta : I(x; y|z = \zeta) < \delta\}$. By Markov's inequality, $\mathbf{P}(z \notin Z) \leq \delta^{-1} I(x; y|z) < \delta$. For every $\zeta \in Z$, there is a coupling $\mathcal{Q}_\zeta \in \Delta(\Delta(X) \times \Delta(X))$ with marginals $p(x|z = \zeta)$ and $p(x|y, z = \zeta)$, such that $\int \|x - y\| \, \mathrm{d}\mathcal{Q}_\zeta(x, y) < \epsilon$. For $\zeta \notin Z$, let $\mathcal{Q}_\zeta \in \Delta(\Delta(X) \times \Delta(X))$ be an arbitrary coupling of $p(x|z = \zeta)$ and $p(x|y, z = \zeta)$. The distribution $\mathcal{Q} = \sum_\zeta \mathbf{P}(z = \zeta)\mathcal{Q}_\zeta$ is a coupling of $p(x|z)$ and $p(x|y, z)$, and

$$\begin{aligned}\int \|x - y\| \, \mathrm{d}\mathcal{Q}(x, y) &= \sum_{\zeta \in Z} \int \|x - y\| \, \mathrm{d}\mathcal{Q}_\zeta(x, y) \\ &< \mathbf{P}(z \in \zeta)\epsilon + \mathbf{P}(\zeta \notin Z) < \epsilon + \delta.\end{aligned}$$

Lemma 3.15 (Information Criterion). *Let A be a finite alphabet. For every $\epsilon > 0$ there is $\delta > 0$ such that for every $n \geq 1$, every $Q \in \Delta(A \times [n])$, every $r > 0$, and every random play $\bar{a} = a_1, \ldots, a_r$, if $\bar{a}$ can be coupled with an $[n]$-valued random sequence $\bar{b} = b_1, \ldots, b_r$ such that*

(i) $\left\| Q - \mathbb{E}[\mathrm{emp}(\bar{a}, \bar{b})] \right\| < \delta$,
(ii) $\frac{1}{r} H(\bar{a}|\bar{b}) > H(x|y) - \delta$, *where* $(x, y) \sim Q$,
(iii) $\frac{1}{r} I(\bar{a}; \bar{b}) < \delta$,

then

$$\bar{d}(\mathrm{p}(x|y), \quad \mathcal{P}_r(\bar{a})) < \epsilon.$$

Note that the conclusion of Lemma 3.15 refers to the distribution of beliefs of an observers who observes only $\bar{a}$ and does not observe the auxiliary random variable $\bar{b}$. Also note that the case of $n = 1$ is equivalent to Lemma 3.14 by setting $\mathcal{P} = \mathcal{P}_r(\bar{a})$.

Proof Since the mapping that maps the distribution of (x, y) to $\mathrm{p}(x|y)$ is continuous as a function form $\Delta(A \times [n])$ to $\Delta\Delta(A)$, we may assume w.l.o.g. that $\mathbb{E}[\mathrm{emp}(\bar{a}, \bar{b})] = Q$. Let $\epsilon > 0$. By (3.2) we can take $\delta > 0$ such that

$$I(a; b|c) < \delta \quad \Rightarrow \quad \bar{d}(\mathrm{p}(a|c), \mathrm{p}(a|b, c)) < \tfrac{1}{2}\epsilon,$$

for any random variables a, b, c, where a takes values in A.

Let t be a random variable uniformly distributed in $[r]$ independently of $(\bar{a}, \bar{b})$. Let $\mathcal{H}_t = a_1, \ldots, a_t$. Note that $(a_t, b_t) \sim Q$, and $\mathcal{P}_r(\bar{a}) = \mathrm{p}(a_t|\mathcal{H}_{t-1}, t)$.

By the triangle inequality

$$\begin{aligned}\bar{d}(\mathrm{p}(a_t|b_t), \mathrm{p}(a_t|\mathcal{H}_{t-1}, t)) \leq\ & \bar{d}(\mathrm{p}(a_t|b_t), \mathrm{p}(a_t|b_t, \mathcal{H}_{t-1}, t)) \\ & + \bar{d}(\mathrm{p}(a_t|b_t, \mathcal{H}_{t-1}, t), \mathrm{p}(a_t|\mathcal{H}_{t-1}, t)).\end{aligned}$$

By the choice of δ, the proof will be concluded if we prove two inequalities:

$$I(a_t; \mathcal{H}_{t-1}, t|b_t) < \delta, \tag{3.3}$$
$$I(a_t; b_t|\mathcal{H}_{t-1}, t) < \delta. \tag{3.4}$$

For (3.3):

$$\begin{aligned}I(a_t; \mathcal{H}_{t-1}, t|b_t) &= H(a_t|b_t) - H(a_t|\mathcal{H}_{t-1}, t, b_t) \\ &\leq H(a_t|b_t) - H(a_t|\mathcal{H}_{t-1}, t, \bar{b}) \\ &= H(a_t|b_t) - \frac{1}{r} H(\bar{a}|\bar{b}) \\ &< \delta\end{aligned}$$

where the last inequality is provided by condition (ii) of the lemma.

For (3.4):

$$\begin{aligned}I(a_t; b_t|\mathcal{H}_{t-1}, t) &= H(a_t|\mathcal{H}_{t-1}, t) - H(a_t|\mathcal{H}_{t-1}, t, b_t) \\ &\leq \frac{1}{r}[H(\bar{a}) - H(\bar{a}|\bar{b})] \\ &= \frac{1}{r} I(\bar{a}; \bar{b}) \\ &< \delta\end{aligned}$$

where the last inequality is provided by condition (iii) of the lemma. □

3.5 Main constructive lemma

The next lemma plays an important role in the proofs of Theorems 2.2, 2.3, and 2.4.

Lemma 3.16 *Let $\mathcal{P} \in \Delta\Delta(A)$ be a finitely supported rational distribution over rational beliefs. For every $\epsilon > 0$, there is $l_0 \in \mathbb{N}$, such that for every $l \geq l_0$ and every De Bruijn automaton scheme for agent 1*

$$\Xi = \left\langle \mathcal{P}, l, \left\{ x(k), L_k, (s_t(k), z_t(k), a_t^2(k))_{t=1}^{|L_k|}, G_k \right\}_{k=1}^{\infty} \right\rangle,$$

the following holds:

With the notation of Definition 3.9, denote the induced joint strategy $\tau_k \in \Delta(\Sigma^1(m_k), \Sigma^2)^{r_k}$. *Suppose* $C = \liminf_{k\to\infty} \frac{|L_k|}{m_k} > 0$. *There is* $r_0 \in \mathbb{N}$ *such that for every* $r \geq r_0$, *there is* $k \geq 1$ *such that* $r' = \left\lfloor \frac{r}{r_k} \right\rfloor \cdot r_k \geq (1-\epsilon)r$ *and the concatenation of* $\left\lfloor \frac{r}{r_k} \right\rfloor$ *independent copies of* τ_k, $\tau \in \Delta(\Sigma^1(m), \Sigma^2)^{r'}$ $(m = \left\lfloor \frac{r}{r_k} \right\rfloor \cdot m_k)$, *satisfies*

$$\frac{\bar{H}(P)}{C} + \epsilon \geq \frac{m \log m}{r}$$

and

$$\bar{d}(\mathcal{P}, \mathcal{P}_r(\tau)) < \epsilon.$$

Proof Let $\mathcal{P} = \sum_{i=1}^{n} q_i \delta_{Q_i}$ be a finitely supported rational distribution over rational beliefs, and let $\epsilon > 0$. Let $\delta = \delta(\frac{\epsilon}{3}) > 0$ be given by Lemma 3.15. Let l be a common denominator of $\mathcal{P}$, such that $f(l)$ of Lemma 3.13 (iv) is less than δ.

Let

$$\Xi = \left\langle \mathcal{P}, l, \left\{ x(k), L_k, (s_t(k), z_t(k), a_t^2(k))_{t=1}^{|L_k|}, G_k \right\}_{k=1}^{\infty} \right\rangle$$

be De Bruijn automaton scheme for agent 1.

Lemma 3.13 provides strategies $\tau_k \in \Delta(\Sigma^1(m_k) \times \Sigma^2)^{r_k}$. The same lemma ensures that $\limsup_{k\to\infty} \frac{m_k \log m_k}{r_k} \leq \frac{\bar{H}(\mathcal{P})}{C}$. Lemma 3.13 (iii)–(iv) and the choice of l ensure that the antecedents of Lemma 3.15 hold w.r.t. δ, for every k large enough. The choice of δ was made such that Lemma 3.15 guarantees that $\bar{d}(\mathcal{P}, \mathcal{P}_{r_k}(\tau_k)) < \frac{\epsilon}{3}$, for every k large enough.

We next utilise Lemma 3.6 to extends the construction to strategies that induce a play of any length $r > r_0$, for some $r_0 > 0$. The antecedents of Lemma 3.6

(i) $\sup_{k\in\mathbb{N}} r_{k+1} = \infty$, and
(ii) $\sup_{k\in\mathbb{N}} \frac{r_{k+1}}{r_k} < \infty$

hold since, by Definition 3.9,

$$\begin{aligned}(k-1)lL_k &\leq r_k \leq klL_k,\\ (C - o(1))m_k &\leq L_k \leq (\left|A^2\right| - 1)m_k,\\ m_k &= l|Y|^k,\\ &\text{and } |Y| \text{ depends only on } l \text{ and } \mathcal{P}.\end{aligned}$$

It follows that there is $r_0 \geq 0$ such that for every $r \geq r_0$, there is $k \geq 1$ such that $r' = \left\lfloor \frac{r}{r_k} \right\rfloor \cdot r_k \geq (1-\epsilon)r$ and the concatenation of $\left\lfloor \frac{r}{r_k} \right\rfloor$ independent copies of τ_k, $\tau \in \Delta(\Sigma^1(m), \Sigma^2)^{r'}$ $(m = \left\lfloor \frac{r}{r_k} \right\rfloor \cdot m_k)$ satisfies

$$\frac{m \log m}{r} \leq \frac{m \log m}{r'} < \frac{\bar{H}(\mathcal{P})}{C} + \frac{\epsilon}{3},$$

and

$$\bar{d}(\mathcal{P}, \mathcal{P}_{r'}(\tau)) < \limsup_{k \to \infty} \bar{d}(\mathcal{P}, \mathcal{P}_{r_k}(\tau_k)) + \frac{\epsilon}{3} < \frac{\epsilon}{3} + \frac{\epsilon}{3}.$$

The same τ satisfies

$$\bar{d}(\mathcal{P}, \mathcal{P}_r(\tau)) < \bar{d}(\mathcal{P}, \mathcal{P}_{r'}(\tau)) + \bar{d}(\mathcal{P}_{r'}(\tau), \mathcal{P}_r(\tau)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3},$$

as desired. □

3.6 One automaton

The stage is set for proving Theorem 2.2.

Proof of Theorem 2.2 Let $\mathcal{P} \in \Delta\Delta(A)$ and $\epsilon > 0$. Since the function $\bar{H}(\cdot)$ is continuous and the set of distributions with finite support and rational coefficients is dense in $\Delta\Delta(A)$, we may assume w.l.o.g. that $\mathcal{P}$ is a rational distribution over rational beliefs. Let $l_0 = l_0(\epsilon)$ be given by Lemma 3.16. Let $l \geq l_0$ be a common denominator of $\mathcal{P}$.

We define a De Bruijn automaton scheme for agent 1

$$\Xi = \left\langle \mathcal{P}, l, \left\{ x(k), L_k, (s_t(k), z_t(k), a_t^2(k))_{t=1}^{|L_k|}, G_k \right\}_{k=1}^{\infty} \right\rangle$$

with

- $x(k) = x_1, \ldots, x_{m_k}$ an arbitrary compound De Bruijn sequence as specified by Definition 3.9,
- $|L_k| = (|A^2| - 1)m_k$,
- $z_t(k) = kl - 1$ for every $t \in [|L_k|]$,
- $\left\{(s_t(k), a_t^2(k))\right\}$ an arbitrary enumeration of all pairs (s, a^2) such that $s \in [m_k]$ and $a^2 \neq x_s^2(k)$,
- G_k the entire symmetry group on $[|L_k|]$.

Since $\frac{|L_k|}{m_k} = |A^2| - 1$, Lemma 3.16 ensures that there is $r_0 \in \mathbb{N}$, such that for every $r \geq r_0$, there is $m \in \mathbb{N}$ and $\tau \in \Delta(\Sigma^1(m) \times \Sigma^2)$ such that

$$\frac{\bar{H}(P)}{C} + \epsilon \geq \frac{m \log m}{r}$$

and

$$\bar{d}(\mathcal{P}, \mathcal{P}_r(\tau)) < \epsilon.$$

□

3.7 One automaton, one sequence

In this section we prove Theorem 2.4. Theorem 2.4 refers only to $\mathcal{P}^1_r(\tau)$, and so we can take any coupling $\mathcal{P} \in \Delta\Delta(A)$ such that $\mathcal{P}^1 = \mathcal{P}^1_r(\tau)$; therefore $H(\mathcal{P}) = H(\mathcal{P}_1)$. In particular, $\mathcal{P}$ can be taken such that $P^2 = \delta_{a^2}$, for some $a^2 \in A^2$ ($\mathcal{P}$ a.s.).

We prove Theorem 2.4 with $C(\mathcal{P}^1) = \frac{|A^2|-1}{\bar{H}(\mathcal{P})} + 0.00001$, assuming w.l.o.g. that $\bar{H}(\mathcal{P}) > 0$. Otherwise, we can approximate $\mathcal{P}$ by a pure periodic sequence with period $\log m$ and consider any r sufficiently large. By continuity, we assume w.l.o.g. that $\mathcal{P}$ is finitely supported and has rational coefficient.

Given $\epsilon > 0$ and a rational distribution over rational beliefs $\mathcal{P} \in \Delta\Delta(A)$ with $\mathcal{P}^2 = \delta_{\delta_{a^2}}$. Let l_0 be given by Lemma 3.16 and let $l \geq l_0$ be a common denominator of $\mathcal{P}$.

Define a De Bruijn automaton scheme for agent 1

$$\Xi = \left\langle \mathcal{P}, l, \left\{ x(k), L_k, (s_t(k), z_t(k), a^2_t(k))_{t=1}^{|L_k|}, G_k \right\}_{k=1}^{\infty} \right\rangle$$

with

- $x(k) = x_1, \ldots, x_{m_k}$ an arbitrary compound De Bruijn sequence as specified by Definition 3.9,
- $|L_k| = (|A^2| - 1)m_k$,
- $z_t(k) = kl - 1$ for every $t \in [|L_k|]$,
- $\left\{(s_t(k), a^2_t(k))\right\}$ an enumeration of all pairs $(s, a^2) \in [m_k] \times A^2 \setminus \{a^2\}$, such $a^2_t(k)$ is $(|A^2| - 1)$-periodic (in t),
- G_k is the group of all the permutations π on $[|L_k|]$ such that $\pi(t) = t \mod |A^2| - 1$, for all $t \in [|L_k|]$.

We show is that G_k is large enough, so that $\sup_{k\in\mathbb{N}} \left(\frac{|L_k|!}{|G_k|}\right)^{\frac{1}{|L_k|}} < \infty$, and so Ξ is a De Bruijn automaton scheme. Indeed,

$$|G_k| = \left(\frac{|L_k|}{|A^2| - 1}!\right)^{|A^2|-1}$$

and therefore,

$$\left(\frac{|L_k|!}{|G_k|}\right)^{\frac{1}{|L_k|}} = \binom{|L_k|}{\frac{|L_k|}{|A^2|-1}, \ldots, \frac{|L_k|}{|A^2|-1}}^{\frac{1}{|L_k|}} \leq (|A^2| - 1)^{|L_k|\frac{1}{|L_k|}} = |A^2| - 1.$$

Let $\tau_k \in \Delta(\Sigma^1(m_k) \times A^2_p(r_k))^{r_k}$ be the induced mixed strategies, i.e., the uniform distribution over $\{(\sigma_{\pi.\Xi}(k), \bar{a}_{\pi.\Xi}(k)) : \pi \in G_k\}$ (as prescribed by Lemma 3.13).

We claim that τ^2_k is pure and that its period is in fact $kl(|A^2| - 1)$ which divides r_k. The choice of the jump transition $\left\{(s_t(k), a^2_t(k))\right\}$ was made such that the period of the induced play $\bar{a}^2_{\Xi}(k)$ is $kl(|A^2| - 1)$ which divides r_k since $r_k = kl|L_k|$ and $|L_k| = (|A^2| - 1)m_k$. The definition of G_k is such that $\bar{a}^2_{\pi.\Xi}(k) = \bar{a}^2_{\Xi}(k)$, for all

$\pi \in G_k$; therefore τ_k^2 is pure. Since $m_k = l|Y|^k$, we can take $C = \frac{|A^2|-1}{\bar{H}(\mathcal{P}^1)} + 0.00001$ and get $\bar{a}_{\Xi}^2(k) \in A_p^2(C \log m_k)$, for any k large enough.

Lemma 3.16 provides $r_0 \in \mathbb{N}$ such that for any $r \geq r_0$ there are $m \in \mathbb{N}$ and $\tau \in \Delta(\Sigma^1(m), \Sigma^2)$ such that

$$\frac{\bar{H}(P)}{C} + \epsilon \geq \frac{m \log m}{r}$$

and

$$\bar{d}(\mathcal{P}, \mathcal{P}_r(\tau)) < \epsilon.$$

Furthermore, τ is the concatenation independent copies of τ_k, for some $k \in \mathbb{N}$, and m is a multiple of m_k; therefore τ^2 is pure and periodic with period $kl(|A^2|-1)$ which is at most $C \log(m_k) \leq C \log(m)$, for all k large enough. For finitely many small values of k, choosing r_0 large enough allows us to choose m so large that $kl(|A^2|-1) \leq C \log(m)$. □

3.8 Pairs of automata

In this section we prove Theorem 2.3. Recall that it is assumed throughout that $|A^1|, |A^2| \geq 2$.

The set of beliefs $\Delta(A)$ is divided into two regions:

$$\Delta_I(A) = \{Q \in \Delta(A) : \text{"}Q\text{ is supported on either one row or one column"}\},$$
$$\Delta_{II}(A) = \Delta(A) \setminus \Delta_I(A).$$

Every $\mathcal{P} \in \Delta\Delta(A)$ can be (uniquely) represented as $\lambda \mathcal{P}_I + (1-\lambda)\mathcal{P}_{II}$, where $\mathcal{P}_{I,II} \in \Delta\Delta_{I,II}(A)$ and $0 \leq \lambda \leq 1$. The constant $C(\mathcal{P})$ of Theorem 2.3 is defined as $\lambda C_I(\mathcal{P}_I) + (1-\lambda)C_{II}(\mathcal{P}_{II})$, with $C_{I,II} \colon \Delta\Delta_{I,II}(A) \to \mathbb{R}_+$ defined by

$$C_I(\mathcal{P}) = \frac{\bar{H}(\mathcal{P})}{n-1}, \qquad \left(\text{where } n = \min\left\{|A^1|, |A^2|\right\}\right)$$
$$C_{II}(\mathcal{P}) = \frac{\bar{H}(\mathcal{P})}{\bar{D}(\mathcal{P})}.$$

Where, $\bar{D}(\mathcal{P}) = \int D(Q)\, \mathrm{d}\mathcal{P}(Q)$ and $D \colon \Delta(A) \to [0, n-1]$ is a continuous function whose zeros are exactly $\Delta_I(A)$. The definition of D is deferred to Sect. 3.8.2.

The following lemma allows us to consider each one of the cases $\mathcal{P} \in \Delta\Delta_I(A)$ and $\mathcal{P} \in \Delta\Delta_{II}(A)$ separately.

Lemma 3.17 *Let* $\mathcal{P}_1, \mathcal{P}_2 \in \Delta\Delta(A)$, $C_1, C_2 \geq 0$, *and* $0 \leq \lambda \leq 1$. *Suppose that for every* $\epsilon > 0$ *and every* $l \in \{1, 2\}$ *there exists* $r_0 \in \mathbb{N}$ *such that for every* $r \geq r_0$ *there exist* $r' \leq r$, $m \in \mathbb{N}$, *and* $\tau \in \Delta(\Sigma^1(m) \times \Sigma^2(m))^{r'}$ *such that*

$$\frac{r'}{r} \geq 1 - \epsilon,$$
$$\frac{m \log m}{r'} \leq C_l + \epsilon,$$

and

$$\bar{d}(\mathcal{P}_l, \mathcal{P}_{r'}(\tau)) < \epsilon.$$

Then, for every $\epsilon > 0$ there exists $r_0 \in \mathbb{N}$ such that for every $r \geq r_0$ there exist $r' \leq r$, $m \in \mathbb{N}$ and $\tau \in \Delta(\Sigma^1(m) \times \Sigma^2(m))^{r'}$ such that

$$\frac{r'}{r} \geq 1 - \epsilon,$$
$$\frac{m \log m}{r'} \leq \lambda C_1 + (1 - \lambda) C_2 + \epsilon,$$

and

$$\bar{d}(\lambda \mathcal{P}_1 + (1 - \lambda)\mathcal{P}_2, \mathcal{P}_{r'}(\tau)) < \epsilon.$$

Proof Let $\epsilon > 0$ and $\lambda \in (0, 1)$. Set $\lambda_1 = \lambda$ and $\lambda_2 = 1 - \lambda_1$, and $\mathcal{P} = \lambda_1 \mathcal{P}_1 + \lambda_2 \mathcal{P}_2$. Let $r_0 > 0$ be large enough so that for every $r \geq r_0$ and $l \in \{1, 2\}$, there are $r_l \leq \lambda_l r$, $m_l \in \mathbb{N}$, and $\tau_l \in \Delta(\Sigma^1(m_l) \times \Sigma^2(m_l))^{r_l}$ such that

$$\frac{r_l}{\lambda_l r} \geq 1 - \epsilon,$$
$$\frac{m_l \log m_l}{r_l} \leq C_l + \epsilon,$$

and

$$\bar{d}(\mathcal{P}_l, \mathcal{P}_{r_l}(\tau_l)) < \epsilon.$$

Let $r' = r_1 + r_2$ and $m = m_1 + m_2$. Let τ be the concatenation of τ_1 and τ_2. We have,

$$\tau \in \Delta(\Sigma^1(m) \times \Sigma^2(m))^{r'},$$
$$\frac{r'}{r} \geq 1 - \epsilon,$$

and

$$\bar{d}(\mathcal{P}, \mathcal{P}_{r'}(\tau)) < \epsilon.$$

It remains to verify

$$\begin{aligned}\frac{m\log m}{r'} &= \sum_{l=1,2}\frac{m_l\log m}{r'} = \sum_{l=1,2}\frac{m_l\log m_l}{r'} + \sum_{l=1,2}\frac{m_l\log\frac{m}{m_l}}{r'}\\ &\le \frac{r}{r'}\sum_{l=1,2}\lambda_l\frac{m_l\log m_l}{r_l} + \frac{m}{r'}\sum_{l=1,2}\frac{m_l}{m}\log\frac{m}{m_l}\\ &\le \frac{1}{1-\epsilon}\left[\sum_{l=1,2}\lambda_l C_l + \epsilon\right] + \frac{m}{r'}H\left(\frac{m_1}{m},\frac{m_2}{m}\right)\\ &\le \sum_{l=1,2}\lambda_l C_l + \frac{\epsilon}{1-\epsilon}(\lambda_1 C_1 + \lambda_2 C_2 + 1) + o(1).\end{aligned}$$

□

By Lemmas 3.6 and 3.17, it is sufficient to prove the following lemma.

Lemma 3.18 *For every $\epsilon > 0$ and every $\mathcal{P} \in \Delta\Delta_I(A) \cup \Delta\Delta_{II}(A)$ there exist sequences $r_k, m_k \in \mathbb{N}$ and $\tau_k \in \Delta(\Sigma^1(m_k) \times \Sigma^2(m_k))^{r_k}$ satisfying*

$$\sup_{k\in\mathbb{N}} r_k = \infty, \tag{3.5}$$

$$\sup_{k\in\mathbb{N}} r_{k+1}/r_k < \infty, \tag{3.6}$$

$$\limsup_{k\to\infty}\frac{m_k\log m_k}{r_k} \le C(\mathcal{P}) + \epsilon, \tag{3.7}$$

$$\limsup_{k\to\infty}\bar{d}(\mathcal{P}, \mathcal{P}_{r_k}(\tau_k)) < \epsilon. \tag{3.8}$$

3.8.1 $C_I(\mathcal{P})$

In this section we prove Lemma 3.18 in the case $\mathcal{P} \in \Delta\Delta_I(A)$. We partition $\Delta_I(A)$ into finitely many regions. By virtue of Lemma 3.17, we assume w.l.o.g. that $\mathcal{P}$ is supported in one of these regions.

There are $|A^1||A^2|+|A^1|+|A^2|$ regions as follows: Dirac beliefs, beliefs supported on any single row, and the beliefs supported on any single column:

$$\begin{array}{ll}\{\delta_a\} & a\in A,\\ \{Q\in\Delta(\{a^1\}\times A^2) : H(Q)>0\} & a^1\in A^1,\\ \{Q\in\Delta(A^1\times\{a^2\}) : H(Q)>0\} & a^2\in A^2.\end{array}$$

The case $\mathcal{P} = \delta_{\delta_a}$ is simple. Let $b \in \left(A^1\setminus\{a^1\}\right)\times\left(A^2\setminus\{a^2\}\right)$. Let σ_k^1 be an automaton with k states that k-periodically outputs $k-1$ consecutive a^1s followed by a b^1 when it reaches state k. Let σ_k^2 be an automaton that $k+1$-periodically outputs k consecutive a^2s followed by a b^2 at state $k+1$. The transitions (k, b^2) and $(k+1, b^1)$

are both first hit at time $k(k+1)$; therefore this pair of automata is $r_k = k(k+1)$ concatenable. Lemma 3.18 holds with $m_k = k+1$.

It remains to consider the case $\mathcal{P} \in \Delta\Delta(A^1 \times \{a^2\})$ with $\bar{H}(\mathcal{P}) > 0$ (the case $\mathcal{P} \in \Delta\Delta(\{a^1\} \times A^2)$ is symmetric). The construction builds on the construction in the proof of Theorem 2.4. We consider m_k, r_k, L_k and the automata $\sigma^1(k)$ and $\sigma^2(k)$, as given in the proof of Theorem 2.4.

We construct an $(r_k + kl)$-concatenable pair of automata with $m_k + kl$ states that generate the same sequence as $(\sigma^1(k), \sigma^2(k))$ in the first kl steps. Since $kl \ll m_k, r_k$, the conditions of Lemma 3.18 are satisfied.

Agent 1's automaton is the concatenation of $\sigma^1(k)$ with an oblivious automaton with kl states. The oblivious automaton starts at state 1 and moves to state 2 and then 3, and so on until it reaches state kl, while always outputting some fixed action $\mathbf{a^1} \in A^1$. Note that L_k does not include a sequence where agent 1 always plays $\mathbf{a^1}$.

The $kl(|A^2|-1)$-periodic oblivious automaton σ^2 is paired with an automaton that looks for the ending sequence of kl consecutive $\mathbf{a^1}$ actions. Formally, suppose the states of σ^2, $[kl(|A^2|-1)]$, are visited in increasing order $1, 2, \ldots, kl(|A^2|-1)$. The states of the new automaton are $[kl(|A^2|-1)] \times \{0, 1\}$. The transitions are of the form $(i, b, a^1) \mapsto (i+1 \bmod kl(|A^2|-1), x)$, where $x = 1$ if and only if either $i = kl(|A^2|-1)$, or $b = 1$ and $a^1 = \mathbf{a^1}$.

After r_k steps the automaton of agent 1 reaches the final transition of σ_k^1 and moves to the initial state of its second automaton. At the same time agent 2's automaton is at state $(1, 1)$. In the next kl steps agent 1 plays $\mathbf{a^1}$ which result in both agents reaching certain states for the first time: agent 2—state $(kl, 1)$, and agent 1—the last state of its second automaton. Therefore, our construction is $r_k + kl$ concatenable.

3.8.2 $C_{II}(\mathcal{P})$

In this section we prove Lemma 3.18 in the case $\mathcal{P} \in \Delta\Delta_{II}(A)$. Recall that the definition of $C(\mathcal{P}) = C_{II}(\mathcal{P})$ depends on an appropriate definition of a function $D\colon \Delta_{II}(A) \to (0, n-1]$. We begin by defining[3] the function D.

We think of A as the complete bipartite graph with colour sets A^1, A^2. A set of action profiles $J \subset A$ is called a *matching* if for every $(a^1, a^2), (b^1, b^2) \in J$, $a^1 = b^1$ if and only if $a^2 = b^2$.

Definition 3.19 We define $D\colon \Delta(A) \to [0, n-1]$ by

$$D(Q) = \max_{\text{matching } J} |J|(|J|-1) \min_{a \in J} Q(a).$$

Note that D is continuous and its range is indeed $[0, n-1]$. The maximum, $n-1$, is attained when Q is a uniform distribution on a perfect matching.[4,5] The mini-

[3] The function D is not the largest possible. Its concavification $cav\, D$ is also possible as discussed in Sect. 3.8.3.

[4] A perfect matching is a matching of size n.

[5] Section 3.8.3 suggests that D can be replaced by $cav\, D$ which implies that $n-1$ can be attained whenever the two marginals of Q, Q^1 and Q^2, are uniform distributions supported on sets of size n.

mum, 0, is attained exactly when Q is supported on either a single row or a single column.

The construction is based on the existence of a pair of De Bruijn automaton schemes, one for each agent, such that the two schemes induce exactly the same play.

Definition 3.20 A *De Bruijn bi-automata scheme* is a tuple

$$\Phi = \left\langle \mathcal{P}, l, \left\{ x(k), L_k, (s_t^1(k), s_t^2(k), z_t(k))_{t=1}^{|L_k|}, G_k \right\}_{k=1}^{\infty} \right\rangle$$

such that for each $i \in \{1, 2\}$,

$$\Phi_i = \left\langle \mathcal{P}, l, \left\{ x(k), L_k, (s_t^i(k), z_t(k), x^{-i}_{s_t^{-i}(k)}(k))_{t=1}^{|L_k|}, G_k \right\}_{k=1}^{\infty} \right\rangle$$

is a De Bruijn automaton scheme for agent i.

The definition of a De Bruijn bi-automata scheme requires that both Φ_1 and Φ_2 have the same L_k and G_k; therefore they induce exactly the same play $\bar{a}_{\Phi_1}(k) = \bar{a}_{\Phi_2}(k)$. This requirement is quite demanding. It says that along the play the two automata always either perform a $+1$ transition together or a jump transition together. I.e., if at some stage the automata are at states s and t respectively, then either $x_s = x_t$, or $x_s^1 \neq x_t^1$ and $x_s^2 \neq x_t^2$. In the latter case we say that the pair (s, t) is *good*. Identifying large sets of good pairs of states (that satisfy some further independence property) will be crucial in our construction of a bi-automaton scheme. The pair of random automata induced by Φ are correlated by using the same $\pi \in G_k$ for both $\sigma^1_{\pi.\Phi_1}(k)$ and $\sigma^2_{\pi.\Phi_2}(k)$.

Let $\mathcal{P} \in \Delta\Delta_{II}(A)$. By continuity, we may assume w.l.o.g. that $\mathcal{P} = \sum_i q_i \delta_{Q_i}$ is finitely supported rational distribution over rational believes. Let l be an arbitrary common denominator of $\mathcal{P}$. By Lemma 3.16, it is sufficient for the proof of Lemma 3.18 to construct a De Bruijn bi-automata scheme

$$\Phi = \left\langle \mathcal{P}, l, \left\{ x(k), L_k, (s_t^1(k), s_t^2(k), z_t(k))_{t=1}^{|L_k|}, G_k \right\}_{k=1}^{\infty} \right\rangle$$

with

$$\liminf_{k\to\infty} \frac{|L_k|}{m_k} \geq \bar{D}(\mathcal{P}) - \epsilon(l),$$

where $\epsilon(l) \to 0$, as $l \to \infty$.

Denote $l_j = q_j l$, a common denominator of Q_j. Let $Y = \times_j T^{Q_j}(l_j)$, and $x(k) = x_1, \ldots, x_{m_k}$ an arbitrary compound De Bruijn sequence of order k over the alphabet Y. The group G_k is the entire symmetry group on $[|L_k|]$. It remains to specify $|L_k|$, $s_t^1(k)$, $s_t^2(k)$, and $z_t(k)$.

In what follows we sometimes suppress the index k when it causes no confusion. For a state $s \in [m_k]$, let $\{lt+1, \ldots, lt+l\} \ni s$ be the unique non-overlapping interval of l consecutive states containing s. We consider the chunk of x that begins $(k-1)l$ places before that interval and ends at s. That is, let $u = u(s)$ be the unique integer such that $(k-1)l \leq u < kl$ and $s = u + 1 \mod l$. We call the chunk of the De Bruijn sequence $x_{[s-u,s-1]} := (x_{s-u}, \ldots, x_{s-1})$ the *stem* of s, denoted $\operatorname{stem}(s)$. We call the state $s-u$ the *origin* of s, denoted $\operatorname{orig}(s)$.

For states $s, t \in [m_k]$ corresponding to elements of the compound De Bruijn sequence $x_s = (a^1, a^2)$ and $x_t = (b^1, b^2)$, we say that (s,t) is a *good pair of states* if

(i) $\operatorname{stem}(s) = \operatorname{stem}(t)$, and
(ii) $a^1 \neq b^1$ and $a^2 \neq b^2$.

The *played action* at (s,t) is (a_1, b_2) and we use the notations:

$$\begin{aligned} \operatorname{act}(s,t) &= (a^1, b^2), \\ \operatorname{act}^1(s) &= a^1, \\ \operatorname{act}^2(t) &= b^2. \end{aligned}$$

A set of good pairs of states X is *independent* if for every $(s,t), (s',t') \in X$

$$\begin{aligned} s = s' &\Rightarrow \operatorname{act}^2(t) \neq \operatorname{act}^2(t'), \quad \text{and} \\ t = t' &\Rightarrow \operatorname{act}^1(s) \neq \operatorname{act}^1(s'). \end{aligned} \tag{3.9}$$

In graph theoretic terminology, X is an independent set of vertices in an auxiliary graph whose vertices are the good pairs of states and whose edges are given by (3.9).

From an independent set of good pairs of states of size L, $\left\{(s_t^1, s_t^2)\right\}_{t=1}^{L}$, we construct the remaining components of Φ by setting $|L_k| = L$, and $z_t = u(s_t^1) = u(s_t^2)$, for all $t = 1, \ldots, L$. Condition (3.9) ensures that the fifth item of Definition 3.9 is satisfied.

Therefore, it remains to find an independent set of size $\geq (\bar{D}(\mathcal{P}) - \epsilon(l))m_k$ for every l and every k large enough, where $\epsilon(l) \to 0$, as $l \to \infty$.

With a tolerable abuse of notation define

$$D(J, Q) = |J|(|J| - 1) \min_{a \in J} Q(a),$$

for a matching J and a distribution $Q \in \Delta(A)$. Recall that

$$D(Q) = \max_J D(J, q).$$

Let J_j^* be a matching such that $D(Q_j) = D(J_j^*, Q_j)$.

It will be convenient to consider two partitions of the states: a coarser one and a finer one. The coarser partition is made by grouping together states with the same stem. We say the two states $s, t \in [m_k]$ are equivalent if they have the same stem. Namely, $s \sim t$ if $\operatorname{stem}(s) = \operatorname{stem}(t)$. The equivalence class of $s \in [m_k]$ is denoted $\langle s \rangle$. The finer partition is given by further considering the action x_s. For $a \in A$ and $J \subset A$, we define

$$\langle s\rangle_a = \{t : t \sim s,\ x_t = a\}, \text{ and}$$
$$\langle s\rangle_J = \bigcup_{a\in J} \langle s\rangle_a .$$

These partitions induce conditional distributions $q(\cdot|s) \in \Delta(A)$ defined by

$$q(a|s) = \frac{|\,\langle s\rangle_a\,|}{|\,\langle s\rangle\,|}.$$

Denote $L_j = l_1 + \cdots + l_j$. Note that if $s = i \mod l$, $i \in [L_j] \setminus [L_{j-1}]$, then $q(\cdot|s)$ depends only on $x_{[s-i+L_{j-1}+1,s-1]}$. Explicitly,

$$q(a|s) = \frac{|\{y \in T^{Q_j}(l_j) : y_{[1,i-L_{j-1}-1]} = x_{[s-i+L_{j-1}+1,s-1]},\ y_i = a\}|}{|\{y \in T^{Q_j}(l_j) : y_{[1,i-L_{j-1}-1]} = x_{[s-i+L_{j-1}+1,s-1]}\}|}.$$

Explanation: the states in $\langle s\rangle$ correspond to blocks of x whose prefix is stem(s), which are exactly all the sequences in Y^k beginning with stem(s). Since the first $k-1$ Y-words of these sequences are determined by stem(s), we have

$$\begin{aligned}|\,\langle s\rangle\,| &= |\{y \in Y : y_{[1,i-1]} = x_{[s-i+1,s-1]}\}| \\ &= |\{y \in T^{Q_j}(l_j) : y_{[1,i-L_{j-1}-1]} = x_{[s-i+L_{j-1}+1,s-1]}\}| \times |\textstyle\bigtimes_{j'>j} T^{Q_{j'}}(l_{j'})|,\end{aligned}$$

and similarly,

$$\begin{aligned}|\,\langle s\rangle_a\,| &= |\{y \in Y : y_{[1,i-1]} = x_{[s-i+1,s-1]},\ y_i = a\}| \\ &= |\{y \in T^{Q_j}(l_j) : y_{[1,i-L_{j-1}-1]} \\ &= x_{[s-i+L_{j-1}+1,s-1]},\ y_i = a\}| \times |\textstyle\bigtimes_{j'>j} T^{Q_{j'}}(l_{j'})|.\end{aligned}$$

For a state $s \in [m_k]$, let $q(s) = q(\cdot|s)$ and let $j(s)$ be the index such that $s \in [L_{j(s)}]\setminus[L_{j(s)-1}] \mod l$. When s is drawn uniformly at random from $[m_k]$, $q = q(s)$ and $j = j(s)$ become random variables (functions of s). We denote $q_l = p(q|j)$, emphasizing the dependence of the distribution of (q, j) on l (and not on k or the choice of x).

Note that q_l is the expected distribution of beliefs of a uniform random sample from Y. Lemma 3.15 implies that $q_l \to \mathcal{P}$ in (as $l \to \infty$). Here, Lemma 3.15 is applied with $r = l$, $\bar{a} \sim$ Unifom(Y), and $\bar{b} = b_1, \ldots, b_r$ being the fixed sequence where b_t is the index j such that $t \in [L_j] \setminus [L_{j-1}]$. Since $\lim_{l\to\infty} \frac{\log|T^{Q_j}(l_j)|}{l_j} = H(Q_j)$, for all j, we have $H(\bar{a}|\bar{b}) = H(\bar{a}) \to \bar{H}(\mathcal{P})$, as $l \to \infty$.

Let $J^* = J^*(s) := J^*_{j(s)}$. Since, for any J, the function $D(J, \cdot)$ is continuous on the compact domain $\Delta(A)$, the proof will be concluded if we construct an independent set of size

$$m_k \mathbb{E}[D(J^*, q)].$$

We turn now to construct an independent set of size $m_k \mathbb{E}[D(J^*, q)]$. For $s \in [m_k]$, let $d(s) = \min_{a \in J^*(s)} |\langle s \rangle_a|$. It suffices to find $d(s)|J^*(s)|(|J^*(s)| - 1)$ independent good pairs of states from $\langle s \rangle_{J^*}$, since by doing so we obtain an independent set of pairs $I_{\langle s \rangle} \subset \langle s \rangle \times \langle s \rangle$ of size $|\langle s \rangle| D(J^*(s), q(s))$. Summing over all the equivalence classes $\{\langle s \rangle : s \in [m_k]\}$ gives an independent set of the desired size,

$$\left| \bigcup_{\langle s \rangle : s \in [m_k]} I_{\langle s \rangle} \right| = \sum_{s \in [m_k]} \frac{|I_{\langle s \rangle}|}{|\langle s \rangle|} = \sum_{s \in [m_k]} D(J^*(s), q(s)) = m_k \mathbb{E}[D(J^*, q)].$$

Fix $s \in [m_k]$. For every $a \in J^*(s)$, let $S_a \subset \langle s \rangle_a$, $|S_a| = d(s)$. For every ordered pair $(a, b) \in J^*(s) \times J^*(s)$, $a \neq b$, let $\varphi_{a,b} : S_a \to S_b$ be a bijection. The following set is an independent set of good pairs all coming from $\langle s \rangle_{J^*(s)}$ whose size is $d|J^*(s)|(|J^*(s)| - 1)$:

$$\bigcup_{(a,b) \in J^*(s) \times J^*(s), a \neq b} \left\{ (t, \varphi_{a,b}(t)) : t \in S_a \right\}.$$

3.8.3 *Improving* C_{II}

We suggest an improvement of C_{II}, by replacing the function D by its concavification $cav\ D$. Our proof that shows that $D(Q)$ is achievable could be slightly modified in order to show that $(cav\ D)(Q)$ is achievable. For the sake of simplicity and clarity of the proof, we chose to prove the slightly weaker statement, while only sketching the proof of the stronger statement.

The quantity $(cav\ D)(Q)$ is the solution of the following linear programme:

$$\begin{aligned} \max \sum_{\text{matching } J} x_J |J|(|J| - 1) \qquad & \text{subject to} \\ \sum_{\substack{\text{matching } J: \\ a \in J}} x_J \leq Q(a) \qquad & \text{for every } a \in A, \\ x_J \geq 0 \qquad & \text{for every matching } J. \end{aligned}$$

As before, the idea is to find an independent set of good pairs of states of size $m_k \mathbb{E}[(cav\ D)(q(s))] - o(m_k)$. Take a random state $s \in [m_k]$. Associate with s a bipartite multi-graph M_s whose colour sets are A^1 and A^2, and the multi-edges between each $a^1 \in A^1$ and $a^2 \in A^2$ correspond to $\langle s \rangle_{(a^1, a^2)}$. Let $\{x_J\}$ be an optimal solution for the linear programme $(cav\ D)(q(s))$. Let $d_J = \lfloor x_J |\langle s \rangle| \rfloor$, for every matching J. Note that $d_J / |\langle s \rangle|$ is a feasible solution for the linear programme, and it is nearly optimal, if $|\langle s \rangle|$ is large. The multi-graph M_s contains d_J copies of each matching J. For every copy of J, take the ordered pairs of states that correspond to any ordered pair of multi-edges in J. Do the same for every J. The union of these collections of pairs of states is an independent set of good pairs of states of size $d_*(s) = \sum_J d_J |J|(|J| - 1)$, with

$$\frac{d_*(s)}{\langle s\rangle\,(cav\ D)(q(s))} \to 1, \quad \text{as } |\,\langle s\rangle\,| \to \infty.$$

Remark 3.21 Since $cav\ D(Q)$ is a linear programme, it could be interesting to find a meaningful interpretation to the dual program, which could possibly lead to tightness results.

4 Values of repeated games

We present a few implications of our main results on the min–max values of repeated games played by finite automata.

We consider a class of three-player repeated games, parameterized by a one-shot strategic-form game, automaton size constraints, and game duration.

Formally, $G = (A = A^1 \times A^2 \times A^3, g : A \to \mathbb{R})$ is a three-player game, where g is the payoff to Player 3. The payoff function extends to $g : \Delta(A) \to \mathbb{R}$ linearly. Given such a game, we define different min–max values depending on whether the team of players 1 and 2 are restricted to pure strategies, or can randomize independently, or play correlated strategies. The correlated min–max is defined as

$$\text{cor min max } G = \min_{\tau^{1,2}\in\Delta(A^1\times A^2)} \max_{\tau^3\in\Delta(A^3)} g(\tau^{1,2}\otimes\tau^3).$$

The pure min–max and max–min are defined as

$$\begin{aligned}\text{pure min max } G &= \min_{a^1\in A^1, a^2\in A^2} \max_{a^3\in A^3} g(a^1, a^2, a^3),\\ \text{pure max min } G &= \max_{a^3\in A^3} \min_{a^1\in A^1, a^2\in A^2} g(a^1, a^2, a^3).\end{aligned}$$

The r-stage repeated version of G is denoted $G_r(m_1, m_2, m_3)$, where each player i is restricted to strategies of automaton size m_i, and the payoff is the average per-stage payoff (or the limiting average, if $r = \infty$). We allow for at most one of the parameters to be infinite. In this case, either all player's strategy sets are finite, or r is finite, which guarantees that the game has a finite strategic form.

Since $G_r(m_1, m_2, m_3)$ is a finite game in strategic form, the three values, cor min max, pure min max and pure max min are well defined. For example

$$\begin{aligned}&\text{cor min max } G_r(m_1, m_2, m_3)\\ &= \min_{\tau^{1,2}\in\Delta(\Sigma^1(m_1)\times\Sigma^2(m_2))} \max_{\tau^3\in\Delta(\Sigma^3(m_3))} \mathbb{E}_{\tau^{1,2,3}} \frac{1}{r}\sum_{t=1}^{r} g(a_t^1, a_t^2, a_t^3).\end{aligned}$$

We study asymptotic properties of the min–max values of $G_r(m_1, m_2, m_3)$ and compare them to the min–max values of G.

In what follows $\{m_k\}$, $\{n_k\}$, and $\{r_k\}$ are sequences of natural numbers. If Players 1 and 2 could implement a random play of r_k independent Q-distributed actions, then they could guarantee, in the r_k-stage repeated game, the value that Q guarantees in the one-shot game. Since implementing an i.i.d. play is not always possible under automaton size constraints, we need a notion of approximation to i.i.d. play that guarantees similar strategic power.

Definition 4.1 For $Q \in (A_1 \times A_2)$, an *approximation of* r_k *independent* Q*-distributed random actions* (*approximate i.i.d. play*, for short) is a sequence of distributions $P_k \in \Delta((A^1 \times A^2)^{r_k})$ such that the expected empirical frequency of beliefs of P_k converges to δ_Q w.r.t. the $\bar{d}$ distance (as $k \to \infty$).

Theorems 2.2, 2.3, and 2.4 provide conditions under which Players 1 and 2 can implement such approximate i.i.d. plays. These conditions translate to three Propositions regarding the correlated min–max value.

Theorem 2.2 provides an implementation of an approximate i.i.d. play in situations where Player 1 is restricted and Player 2 is fully rational.

Proposition 4.2 *If* $\lim_{k\to\infty} r_k = \infty$ *and*

$$\liminf_{k\to\infty} \frac{m_k \log m_k}{r_k} \geq \frac{\log(|A^1 \times A^2|)}{(|A^2| - 1)},$$

then

$$\lim_{k\to\infty} \operatorname{cor\,min\,max} G_{r_k}(m_k, \infty, \infty) = \operatorname{cor\,min\,max} G.$$

When both players, 1 and 2, are restricted, Theorem 2.3 provides a similar result, only that in this case the ratio between r_k and $m_k \log m_k$ depends on the one-shot payoff function.

Proposition 4.3 *There exists a constant* $C > 0$*, that depends on* G*, such that if* $\lim_{k\to\infty} r_k = \infty$ *and*

$$\liminf_{k\to\infty} \frac{m_k \log m_k}{r_k} \geq C,$$

then

$$\lim_{k\to\infty} \operatorname{cor\,min\,max} G_{r_k}(m_k, m_k, \infty) = \operatorname{cor\,min\,max} G.$$

When the actions of Player 2 do not influence the payoff, the correlated min–max of the one-shot game is equal to the (uncorrelated) min–max. By Theorem 2.4, the situation is asymptotically the same w.r.t. the repeated version. Furthermore, Player 2's automaton can be very simple: pure, oblivious, and with just $\mathcal{O}(\log m)$ states.

Proposition 4.4 *If the payoff function* g *does not depend on Player 2's actions, then there is* $C > 0$ *such that if* $\lim_{k\to\infty} r_k = \infty$ *and*

$$\liminf_{k\to\infty} \frac{m_k \log m_k}{r_k} \geq \frac{\log |A^1|}{|A^2| - 1},$$

then[6]

$$\lim_{k\to\infty} \min_{\substack{\tau^1\in\Delta(\Sigma^1(m_k))\\ \sigma^2\in A^2_p(\lceil C\log(m_k)\rceil)}} \max_{\sigma^3\in\Sigma^3} g(\tau^1,\sigma^2,\sigma^3) = \text{cor min max } G.$$

Proofs of Propositions 4.2, 4.3, and 4.4 Let $Q \in \Delta(A^1 \times A^2)$ be a correlated min–max strategy for players 1 and 2 in the game G. By Theorem 2.3, there exist a constant $C = C(Q)$ and correlated strategies $\tau_k \in \Delta(\Sigma^1(m_k) \times \Sigma^2(m_k))$ whose induced play of length $Cm_k \log m_k$ approximates a sequence of independent Q-distributed random variables. This proves Proposition 4.3. Similarly, resorting to Theorems 2.2 and 2.4 proves Propositions 4.2 and 4.4 respectively. □

Conversely to Propositions 4.2, 4.3, and 4.4, Player 3 has a pure strategy that guarantees the pure one-shot-game min–max value when the duration of the game is much more than $m \log m$.

Proposition 4.5 *If* $\frac{m_k \log m_k}{r_k} \to 0$*, as* $k \to \infty$*, then*

$$\lim_{k\to\infty} \text{pure max min } G_{r_k}(m_k, m_k, \infty) = \text{pure min max } G.$$

Proof By Lemma 3.1, $\log|\Sigma^1(m_k) \times \Sigma^2(m_k)| = \mathcal{O}(m_k \log m_k)$; Theorem 1 from Neyman and Okada (2009) implies that if one player is restricted to strategies in a set whose size is sub-exponential in the duration of the game and the other player is unrestricted, then the other player can asymptotically guarantee the pure-min–max value using pure strategies. □

We turn now to settings in which Player 3's automaton size is bounded and the duration of the game r_k is not necessarily finite.

Proposition 4.6 *If* $\frac{\log n_k}{m_k \log m_k} \to 0$*, as* $k \to \infty$*, then*

$$\lim_{k\to\infty} \text{cor min max } G_{r_k}(m_k, m_k, n_k) = \text{cor min max } G.$$

Proof Let $Q \in \Delta(A^1 \times A^2)$ be an optimal correlated strategy for Players 1 and 2 in G. By Lemmata 3.6 and 3.18, there exists $C(Q) > 0$ and $r'_k \sim C(Q)m_k \log m_k$, such that Players 1 and 2 can implement an approximation of a r'_k-periodic sequence of independent Q-distributed random actions. We assume w.l.o.g. that the induced play is stationary, because it can be made stationary by taking the average of r'_k shifts of the play; therefore any window of T consecutive actions, $T \le r'_k$, an approximation of independent Q-distributed random actions.

Divide the duration of the game r_k into time intervals of length r'_k with a possible remainder in the beginning. Namely, intervals $[t_{n+1}] \setminus [t_n]$, where $t_0 = 0$, $t_1 = r_k - r'_k \left\lfloor \frac{r_k}{r'_k} \right\rfloor$ (or 0 if $r_k = \infty$), and $t_n = t_1 + (n-1)r'_k$. Let σ^3 be a pure strategy for player 3

[6] Note that cor min max $G = \min_{x^1\in\Delta(A^1)} \max_{a^3\in A^3} g(x^1, \cdot, a^3)$.

(possibly, a best response). For any $n = 0, 1, \ldots$ let $\sigma^3_{t_n}$ be Player's 3 strategy induced on the game starting at time t_n. The random variable $\sigma^3_{t_n}$ $(n > 0)$ assumes values in a set of size n_k depending on the state of Player 3's automaton at time t_n, and $\sigma^3_{t_0} = \sigma^3$ is a fixed strategy. By (Peretz 2012, Corollary 4.3) (also, Neyman 2008, pp. 9, 15–16)), the expected average per-stage payoff between time t_n and time $t_{n+1} - 1$ is asymptotically at most the correlated min–max value of G, as k goes to infinity. Since this is true for any n, the expectation of the payoff is asymptotically at most the correlated min–max value of G. □

Conversely to Proposition 4.6, if $m_k \log m_k$ is not large enough compared to $\log n_k$ then Player 3 can beat players 1 and 2.

Proposition 4.7 *For every $C > |A^3|(|A^1| + |A^2|) - 2$, if $\log n_k \geq C m_k \log m_k$ and $r_k \geq n_k \to \infty$, as $k \to \infty$, then*

$$\lim_{k\to\infty} \text{cor min max } G_{r_k}(m_k, m_k, n_k) = \text{pure min max } G.$$

Proposition 4.7 strengthens Proposition 4.5 in that Player 3 needs only be exponentially smarter than Players 1 and 2 and the duration of the game is allowed to be proportional to $m_k \log m_k$. Nevertheless, Proposition 4.5 is stronger in that Player 3 can do with a pure strategy. Conversely, Players 1 and 2 have a winning pure strategy when they are much smarter than Player 3, as shown in Proposition 4.8 below.

Propositions 4.3 and 4.5 together show that around a duration proportional to $m \log m$ a phase transition occurs. The correlated min–max value of the repeated game changes from the correlated to the pure min–max values of the one-shot game.

Proof of Proposition 4.7 The proof is similar to the proof of (Neyman 1997, Theorem 3). Let $\gamma : A^1 \times A^2 \to A^3$ be a best-response function. That is, $g(a^1, a^2, \gamma(a^1, a^2)) = \max_{a^3 \in A^3} g(a^1, a^2, a^3)$, for every $a^1 \in A^1$, $a^2 \in A^2$. The definition of γ extends to strategies in the repeated game recursively by

$$\gamma : \Sigma^1 \times \Sigma^2 \to (A^3)^{\mathbb{N}},$$
$$\gamma_t(\sigma^1, \sigma^2) = \gamma\left(\sigma^{1,2}\left(\gamma_l\left(\sigma^1, \sigma^2\right)_{l=0}^{t-1}\right)\right),$$

where $\sigma^{1,2}$ is the reduced strategy of the team of players 1 and 2 induced by σ^1 and σ^2. Let $X(m) = \{\gamma(\sigma^1, \sigma^2) : \sigma^1 \in \Sigma^1(m),\ \sigma^2 \in \Sigma^2(m)\}$. Since every pair of automata $(\sigma^1, \sigma^2) \in \Sigma^1(m) \times \Sigma^2(m)$ can be regarded as a single automaton with m^2 states, every $x \in X(m)$ can be implemented through an oblivious automaton with m^2 states.

We next construct an automaton for Player 3 with $m^3|X(m)|$ states. By Lemma 3.1, $\log(m^3|X(m)|) \leq (|A^3|(|A^1|+|A^2|)-2)m \log m + o(m \log m)$. The strategy of Player 3 is the following strategy:

(i) Choose $x \in X(m)$ uniformly at random. Play x as long as it best responds to the actions of players 1 and 2.
(ii) Repeat Step (i) $m|X(m)|$ times or until the end of the game.

(iii) Continue arbitrarily.

The implementation of x in Step (i) requires m^2 states, and so implementing $m|X(m)|$ repetitions of Step (i) requires $m^3|X(m)|$ states, as promised.

The probability of "guessing" the best response in each iteration of Step (i) is at least $|X(m)|^{-1}$, therefore the probability getting to Step (iii) is at most $(1-|X(m)|^{-1})^{m|X(m)|} \to 0$, as $m \to \infty$.

It remains to verify the expected payoff. In each iteration of Step (i) there is at most one non-best-response stage. Since the duration of the game is much longer than the number of iterations, and the probability of getting to Step (iii) is negligible, the expected payoff is guaranteed to be asymptotically at least the pure min–max of the one-shot game. □

If players 1 and 2 are much "smarter" than Player 3, they can implement a fixed play that looks like a sequence of optimal correlated mixed actions in the eyes of Player 3.

Proposition 4.8 *If* $\frac{n_k}{m_k} \to 0$, *as* $k \to \infty$, *then*

$$\lim_{k\to\infty} \text{pure min max}\, G_{r_k}(m_k, m_k, n_k) = \text{cor min max}\, G,$$

for any $r_k \geq m_k \log m_k$.

Proof Let $Q \in \Delta(A^1 \times A^2)$ be an optimal correlated strategy for Players 1 and 2 in G. By Lemmas 3.6 and 3.18, there are $C(Q) > 0$, $r'_k \sim C(Q) m_k \log m_k$, and $\tau_k \in \Delta(\Sigma^1(m_k) \times \Sigma^2(m_k))^{r'_k}$ such that $\mathcal{P}_{r'_k}(\tau_k)$ converges to δ_Q (as $k \to \infty$) w.r.t. the $\bar{d}$ metric. We assume w.l.o.g. that r_k is a multiple of r'_k (otherwise we prove the Proposition with a $n_k \ll m'_k \ll m_k$, such that dividing r_k by r'_k leaves a negligible reminder).

Let $\beta : support(\tau_k) \to \Sigma^3(n_k)$ be any function. It suffices to show that for any such β (possibly a best-reply for Player 3) there is at least one $a \in support(\tau_k)$ such that $g_{r_k}(a, \beta(a)) \leq \text{cor min max}\, G + o(1)$. We show a stronger statement

$$\mathbb{E} g_{r_k}(\sigma, \beta(\sigma)) \leq \text{cor min max}\, G + o(1), \tag{4.1}$$

where σ is a random strategy that distributes according to τ_k. We may assume that $r_k = r'_k$, since otherwise we can divide $[r_k]$ into intervals of length r'_k and prove the statement in each one of these intervals.

Inequality (4.1) follows from (Peretz 2012, Corollary 4.3), since $H(\beta(\sigma)) \leq \log(|\Sigma^3(n_k)|) = o(r'_k)$. □

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