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OVERLAPPING GENERATIONS GAMES WITH MIXED STRATEGIES

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This paper proves a Folk Theorem for overlapping generations games in the case where the mixed strategies used by a player are not observable by the others, but only their realizations are public.

1. Introduction. In the theory of repeated games, the study of the set of equilibrium payoffs has led to different Folk Theorems, so called by analogy with the "classical" Folk Theorem, as defined by Aumann and Shapley (1994) and Rubinstein (1994) in 1977.

Consider a repeated game with perfect recall and standard information. We may assume that after every stage, the mixed moves used by the players to randomize their actions are publicly announced, and we say that *mixed strategies are observable.* A more realistic assumption is that only the action actually chosen by a player, realization of his mixed move, is revealed to the others. We then speak of *nonobservable mixed strategies.*

It is quite easy to see that any equilibrium average payoff must belong to the set V of vector payoffs that are superior to every player's minimax payoff in the one shot game—or *individually rational*—and belong to the convex set of feasible payoffs. One aim of Folk Theorems is to look for minimal assumptions under which every payoff in V is an equilibrium average payoff of the repeated game, or may be approximated by such payoffs. The proofs of Folk Theorems all rely on a common idea. Let v be in V, to construct equilibrium strategies leading to an average payoff near v, one first defines the equilibrium path, also called *Main Path*, of pure actions. If some player deviates from the *Main Path*, the others will then use their minimax strategy (in mixed moves) against him during some number of stages to lower his total payoff. When asking for subgame perfection, punishers should have incentives to punish a deviator. If mixed strategies are observable, any player who does not use the minimax strategy against a deviator will be detected and may be punished himself. In the case of nonobservable mixed strategies, the construction is more tricky and relies on different techniques, using the convexity of the set of achievable payoffs or a statistic on the history.

Until now, the cases of infinitely repeated games with observable or nonobservable mixed strategies, of finitely repeated games and overlapping generations games with observable mixed strategies have been explored. Gossner (1995) proved a Folk Theorem for finitely repeated games that extends the Folk Theorem of Benoît and Krishna (1985) from observable mixed strategies to nonobservable mixed strategies. The proof in Gossner (1995) uses a class of games called games with terminal payoffs. These games were introduced by Kandori (1992) to prove a Folk Theorem for overlapping generations games that was later extended by Smith (1992). While both Kandori and Smith assumed that mixed strategies were observable, this paper

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presents a Folk Theorem for overlapping generations games with nonobservable mixed strategies.

In the next section we present a model for overlapping generations games, and recall the results of Kandori and Smith. Our main result, the Folk Theorem for overlapping generations games with nonobservable mixed strategies, is then proved in §3. Section 4 contains some extensions of the model and of the proof.

2. The model.

2.1. The one-shot game G. We consider a normal form game G, with the set of players $I = \{1, 2, \dots, I\}$. Each player $i \in I$ has a finite set of actions A^i and the payoff function is $g : A = \prod_i A^i \to \mathbb{R}^I$. $\Delta(A^i) = S^i$ represents the set of mixed strategies (or mixed moves) for player i, g will also denote the the canonical extension of the payoff function to $S = \prod_i S^i$.

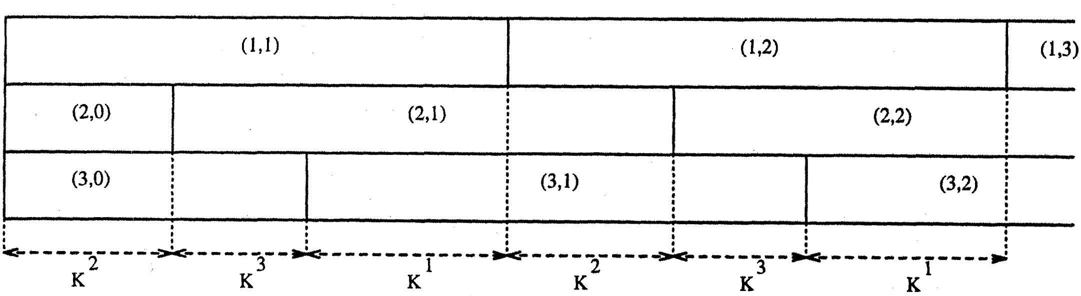
As usual A^{-i} stands for $A^1 \times \cdots \times A^{i-1} \times A^{i+1} \times \cdots \times A^I$, and so for S^{-i}. We choose for every i a minimax strategy in mixed moves against player i, $m_i^{-i} \in S^{-i}$, and a best response m_i^i against m_i^{-i}, so that $m_i = (m_i^{-i}, m_i^i) \in S$.

Without loss of generality, we assume that $g^i(m_i) = 0$ for every i; $F = \operatorname{co} g(A)$ $(= \operatorname{co} g(S))$ and $V = F \cap \mathbb{R}_+^I$ are respectively the set of feasible payoffs and the (closed) set of individually rational and feasible payoffs. For $X \in \mathbb{R}^I$ and $\rho \in \mathbb{R}_+$, $\mathscr{B}(X, \rho)$ denotes the closed ball with center at X and radius ρ.

2.2. Overlapping generations games. Starting with a normal form game G, we define here two different classes of overlapping generations games, or OLGs, with observable or nonobservable mixed strategies.

An OLG is defined through a normal form game which is infinitely repeated from stage $t = 1$ to ∞. The set of players changes over time. Each player plays the game during a finite number of consecutive stages, after which he is replaced by another player of the same type (i.e., same set of actions and same payoff function). His first stage of play is called his birth date. The structure of the OLG is the specification of the set of players at each stage; it defines for every i the birth dates of all players of type i. We will focus on OLGs for which these sequences are periodic (see §4 for an extension of the model). K being a vector of $\mathbb{N}^I$, with $T = \sum_i K^i$, a generic player (i, m), $i \in I$, m in the set $\mathbb{N}^*$ of positive integers, plays (lives) from stage $(m-1)T + \sum_{1<j\le i} K^j + 1$ to $mT + \sum_{1<j\le i} K^j$. Also, there are players $(i, 0)$, $i \neq 1$, that live from stage 1 to $\sum_{1<j\le i} K^j$. Players (i, m) are called players of the mth generation. The players $(i, 0)$ of the 0th generation are of a different nature from the others since they are used to initialize the renewal process of players. Apart from them every player lives T stages. Any player of type i remains the oldest during K^i consecutive stages, such period is called an overlap. K^i is an overlap length and K is the overlap vector.

Structure of an OLG with 3 types of players from $t = 1$ to $t = 2T$:



$G^*(K)$ represents the OLG with structure K and observable mixed strategies. Namely in $G^*(K)$ a history at stage t, $h_t^* \in H_t^*$, is a sequence of t elements of S that describes all the mixed moves in G used by the players until stage t. A strategy of player (i, m) in $G^*(K)$ associates to every history $h_t^* \in H_t^*$ during his life a mixed move in S^i.

However, in the corresponding OLG with nonobservable mixed strategies $G(K)$, a history at stage t, $h_t \in H_t$, describes all the actions played in G until stage t. A strategy in $G(K)$ of player (i, m) specifies at every stage during his life which mixed move he uses knowing the past history.

For simplicity we assume no discount factor on the payoffs, but see §4 for an extension to this case. A subgame perfect equilibrium of $G(K)$ or $G^*(K)$ is in the usual sense a vector of strategies $(\sigma_{i,m})_{i \in I, m \in \mathbb{N}^*}$ such that at every stage of the game and whatever the past history is, a living player (i, m) maximizes his expected future total payoff by using the strategy $\sigma_{i,m}$ if he assumes that every other player (j, n) follows $\sigma_{j,n}$.

Every vector of strategies induces a stream of average expected payoffs. In particular, every stream that is induced by a subgame perfect equilibrium and that takes the same value for any (i, m) and (i, n), $m > 0$ and $n > 0$, will be called a stationary subgame perfect equilibrium payoff. In a stationary equilibrium, all the players of different generations (except of the 0th) but of same type get the same average payoff. Therefore the induced streams of payoffs will be identified with vectors in $\mathbb{R}^I$—thus forgetting the payoffs of players of the 0th generation. $Es(G, K)$ and $E^*s(G, K)$ represent the set of stationary subgame perfect equilibrium payoffs of $G(K)$ and $G^*(K)$ respectively.

If mixed strategies are observable some Folk Theorems hold. The following is due to Kandori (1992):

THEOREM 2.1. *If* $V \cap \mathbb{R}^I_{++} \neq \varnothing$ *then*:

$$\forall \varepsilon > 0, \qquad \exists (K_0^2, \dots, K_0^I) \in \mathbb{N}^{I-1},$$
$$\forall (K^2, \dots, K^I) \gg (K_0^2, \dots, K_0^I), \qquad \exists K_0^1 \in \mathbb{N}, \forall K^1 > K_0^1,$$
$$\forall u \in V, \qquad \mathscr{B}(u, \varepsilon) \cap E^*s(G, K) \neq \varnothing.$$

In words, every payoff in V may be approximated by (subgame perfect) equilibrium payoffs of $G^*(K)$ as soon as there is a long enough delay between the deaths of any pair of players, and if players of a same generation play together during a long time. Basically, players of a generation play a repeated game during K^1 stages, after which their deaths occur, but not simultaneously. Kandori uses this structure to give a reward to the players just before they die, that depends on the past history. Before the player of type 1 dies, he is given his reward during some number of stages. After this, the player of type 2 can be rewarded during K^2 stages without affecting the payoff of the player of type 1 who just died; then comes the turn of the player of type 3, and so on. The OLG is thus similar to a finitely repeated game with terminal payoffs (see Kandori 1992 and Gossner 1995), and Theorem 1 may be viewed as a consequence of a similar result for games with terminal payoffs.

Smith (1992) strengthened this result by proving that a Folk Theorem also holds when all the overlap lengths are independently large:

THEOREM 2.2. *If* $V \cap \mathbb{R}^I_{++} \neq \varnothing$ *then*:

$$\forall \varepsilon > 0, \qquad \exists K_0 \in \mathbb{N}^I, \qquad \forall K \gg K_0,$$
$$\forall u \in V, \qquad \mathscr{B}(u, \varepsilon) \cap E^*s(G, K) \neq \varnothing.$$

3. A Folk Theorem for OLGs with nonobservable mixed strategies. In this section, we present a result equivalent to Theorem 2 in the case where mixed strategies are not observable:

THEOREM A. *If* $V \cap \mathbb{R}^I_{++} \neq \emptyset$ *then*:

$$\forall \varepsilon > 0, \qquad \exists K_0 \in \mathbb{N}^I, \qquad \forall K \gg K_0,$$

$$\forall u \in V, \qquad \mathscr{B}(u, \varepsilon) \cap Es(G, K) \neq \emptyset.$$

To prove this result, we have to take into account the specificity of OLGs and the difficulties implied by the nonobservability of mixed strategies. We will exhibit equilibrium strategies leading to an average payoff near some fixed $u \in V \cap \mathbb{R}^I_{++}$ with an algorithm that defines players' strategies during an overlap. u will be approximated by cycles of payoffs obtained by pure actions during a Main Path, so that any deviation along this path can be detected. If some player deviates, the other players will punish him during some number of stages by using the minimax strategy against him, as commonly done in the proof of Folk Theorems. Nevertheless, there is a problem when dealing with nonobservable mixed strategies: since we want punishers to use a minimax strategy in mixed moves, we must be able to urge them to do so, but we have no possibility of knowing who actually does. We cannot, as Smith and Kandori do, punish finally or immediately any player who is not using the minimax strategy against a deviator.

In the case of infinitely repeated games with little discounting, Fudenberg and Maskin (1986, 1991) make the punishers indifferent between any action by giving them a well-designed continuation payoff after the punishing sequence. To do this they use the fact that every payoff in the convex set V is an average discounted payoff of the infinite repetition of G, see also Neyman (1988) and Sorin (1990). Such a technique may not be used in our case since players' lives are finite, and therefore the set of achievable average payoffs may not be convex.

As done in Gossner (1995) to prove a Folk Theorem for finitely repeated games with nonobservable mixed strategies, we use instead a statistic on the history during a punishing period to work out "good" punishers (called effective ones) and "bad" ones (or noneffective ones); only effective punishers receive a reward at the end of their lives.

3.1. The test functions, their properties. Assume that player i has to be punished during P stages of the game by the other players. At the end of the punishing sequence, we want to compare the actions played by the punishers to some sequence that could have happened if they had played repeatedly their minimax strategy against him. Not only do we want the frequency of actions of each punisher to be near the minimax strategy, but we also want to detect any correlation between the actions of the players.

Consider a punishing sequence history, i.e., a P-tuple $h = (h_{t_0+1}, \dots, h_{t_0+P})$. For every $a \in A$ denote by $n(a)$ the number of occurrences of a in h, and for $a^{-j} \in A^{-j}$ let $n(a^{-j})$ be the number of actions compatible with a^{-j}; $n(a^{-j}) = \sum_{b^j \in A^j} n(a^{-j}, b^j)$.

For $\eta > 0$, a statistic, or test function, on h for player j is given by:

$$\alpha_\eta^{i,j} = \mathbb{1}\left\{\frac{1}{P}\sum_{a \in A}\left|n(a) - n(a^{-j})P(m_i^j = a^j)\right| < \eta\right\}$$

$\mathbb{1}$ is the indicator function and $P(m_i^j = a^j)$ is the probability that j plays a^j while using m_i^j. $\alpha_\eta^{i,j}(h)$ takes the value one whenever the proportion of actions played by j

is near the proportion of the minimax strategy of j against i independently of the actions of the other players; j is then called an effective punisher.

The two following lemmas, proved in Gossner (1995), state that these statistics are well designed for our purpose. Lemma 3.1 shows that if a punisher uses the minimax strategy during some large P periods, he has a high probability of being an effective punisher:

LEMMA 3.1. *For every $\delta > 0$ and $\eta > 0$, there exists P_0 in $\mathbb{N}$ such that for any $P \geq P_0$, if a player $j \neq i$ uses the strategy m_i^j repeatedly during P stages inducing a history h, the probability that $\alpha_\eta^{i,j}(h) = 0$ is less than δ regardless of the strategies used by players other than j.*

Moreover, see that if η is small and all punishers are effective, the average payoff of the punished player during the punishing period is near his minimax payoff:

LEMMA 3.2. *For every $\varepsilon > 0$, there exists $\eta > 0$ such that, for every i and $h = (h_1, h_2, \ldots, h_P) \in H_P$, if for all $j \neq i$, $\alpha_\eta^{i,j}(h) = 1$ then:*

$$\frac{1}{P} \sum_{k=1}^{P} g^i(h_k) < \varepsilon.$$

3.2. *Preliminaries and notations.* Before giving a proof of Theorem A, we would like to make a few remarks and introduce some notations.

First, it should be pointed out that in the statement of Theorem A, K does not depend on $u \in V$, but just on ε. Nevertheless it is enough to prove it for $u \in V \cap \mathbb{R}_{++}^I$ and $\varepsilon > 0$ fixed given the fact that since V is compact, V is included in a finite union of balls of the kind $\mathscr{B}(u_k, \varepsilon)$, $u_k \in V \cap \mathbb{R}_{++}^I$, provided that $V \cap \mathbb{R}_{++}^I$ is nonempty.

Consider such a payoff u and $\varepsilon > 0$. The first step of the proof is to construct two distinct cycles of actions in pure strategies that lead to average payoffs within a distance ε of u.

LEMMA 3.3. *For every $u \in V \cap \mathbb{R}_{++}^I$ and $\varepsilon > 0$, there exist $l \in \mathbb{N}$ and two cycles of l elements of A, $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_l)$ and $\tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_l)$ such that denoting $v = (1/l)\Sigma_t g(\tilde{a}_t)$ and $w = (1/l)\Sigma_t g(\tilde{b}_t)$:*

(i) $\forall i \; v^i \neq w^i$,
(ii) $v \gg 0$ *and* $w \gg 0$,
(iii) $\|v - u\| < \varepsilon$, $\|w - u\| < \varepsilon$.

The proof of this lemma is straightforward. In other words, it says that there exist two cycles $\tilde{a}$ and $\tilde{b}$ that lead to average payoffs v and w near u, such that no player is indifferent between $\tilde{a}$ and $\tilde{b}$ and everyone prefers $\tilde{a}$ and $\tilde{b}$ to his minimax. Let us define the cycles of size $2l$: $\tilde{z} = (\tilde{a}, \tilde{b})$, for every i, $\tilde{z}_+^i = (\tilde{a}, \tilde{a})$ and $\tilde{z}_-^i = (\tilde{b}, \tilde{b})$ if $v^i > w^i$, $\tilde{z}_+^i = (\tilde{b}, \tilde{b})$ and $\tilde{z}_-^i = (\tilde{a}, \tilde{a})$ if $w^i > v^i$. $\tilde{z}_+^i$ is player i's favourite cycle and $\tilde{z}_-^i$ the one he dislikes among $(\tilde{a}, \tilde{a}), (\tilde{b}, \tilde{b})$.

From now on, e^* represents a Nash equilibrium of G.

3.3. The algorithm. With our test functions $\alpha_\eta^{i,j}$ ready as tools telling us which player j is punishing i and which player is not, we can describe the algorithm that defines some subgame perfect equilibria of $G(K)$ with payoff near u. The algorithm is defined on an overlap when i is the oldest player and is then denoted $\mathscr{K}(i)$. It describes a Main Path (MP), which includes the equilibrium path, and says what occurs in case of a deviation from the Main Path.

The algorithm works as follows:

- Most of the overlap is constituted by the Normal Phase (NP), during which a cycle of actions chosen among $(\tilde{a}, \tilde{a})$, $(\tilde{a}, \tilde{b})$, and $(\tilde{b}, \tilde{b})$ is played.

• If some player j deviates "early" in the overlap from the Main Path, the others are asked to punish him during P stages in P(j). After P(j) is finished, we use a set of variables $r(k)$ to keep track of effective and noneffective punishers.

• After the Normal Phase stands the Reward Phase (RP). During (RP), $\tilde{z}^i_+$ is played if i was an effective punisher the last time he was asked to punish another player (case $r(i) = 1$). Otherwise $\tilde{z}^i_-$ is played. Therefore, the reward each player can get at the end of his life is an incentive to be an effective punisher.

• Assume that some player gets a better payoff when the others are rewarded than when they are not. Such a player could deviate to get the benefits from the reward phase of his punishers.

The Compensation Phase (CP), at the beginning of each overlap, is designed to prevent us from any such situation. During a compensation phase, the payoff of the last reward phase is compensated for. This means that if $\tilde{z}^{i-1}_+$ has been played during R turns to reward $i-1$ (case $r' = 1$), $\tilde{z}^{i-1}_-$ is then played during R turns, and conversely. In this way, i is the only player not to be indifferent to the payoff of his own reward phase.

• If it is the case that player i is too old when deviating, there might not be enough time left to punish him.

In the End Phase (EP), he plays a best response during his P last stages of life.

• Another problem with the "old" player i is that after his reward phase, i has no incentive to punish any possible deviator.

This is why if $j \neq i$ deviates "late" in the overlap, the overlap is ended by the play of a Nash equilibrium of G in LD(j). A variable λ is then used to keep track of j, and the punishment of j occurs during the Normal Phase of the next overlap, when his disliked cycle $\tilde{z}^j_-$ is played instead of $\tilde{z}$.

The algorithm starts at $t = 1$ with $i = 2$, $\lambda = 0$, $r(j) = 1$ for every j, $r' = 0$.

The algorithm $\mathscr{K}(i)$:

Main Path (MP):

Compensation Phase (CP):
When $t \in [1, R]$ play $\tilde{z}^{i-1}_+$ if $r' = 0$, $\tilde{z}^{i-1}_-$ if $r' = 1$.

Normal Phase (NP):
When $t \in [R+1, K^i - R - P]$ play $\tilde{z}$ if $\lambda = 0$, else play $\tilde{z}^{\lambda}_-$.

Reward Phase (RP):
When $t \in [K^i - R - P + 1, K^i - P]$ play $\tilde{z}^i_-$ if $r(i) = 0$, $\tilde{z}^i_+$ if $r(i) = 1$, then set $r' = r(i)$.

End Phase (EP):
When $t \in [K^i - P + 1, K^i]$ players other than i play $\tilde{z}$, i plays a best response against them.
Put $\lambda = 0$ and start $\mathscr{K}(i+1)$.

If j deviates from (MP) at stage t_0 go to $P(j)$ or LD(j):

• *If $j \neq i$ and $t_0 \leq K^i - 2P - R$ or $j = i$ and $t_0 \leq K^i - P$:*

Punishment of j P(j)
Play in G during P stages and redefine:
$r(k) = \alpha_{\eta}^{j,k}(h_{t_0+1}, \ldots, h_{t_0+P})$ *for all* $k \neq j$
and keep $r(j)$ *unchanged.*
then go back to (MP).

- *If* $j \neq i$ *and* $t_0 > K^i - 2P - R$:

Late Deviations of j LD(j)
Put $\lambda = j$, *play* e^* *until* $t = K^i$, *and start* $\mathscr{K}(i+1)$.

To simplify the notations, we put $t = 1$ at each first stage of an overlap, and we identify types of players $i = I + 1$ and $i = 1$. In P(j), "go back to (MP)" means "play what is told by (MP) at stage $t_0 + P + 1$." To play a cycle means to play successively all the actions of the cycle, then to start again (to play $\tilde{z}$ is to play $\tilde{z}_k$ at every stage $t = k$ [mod l]).

3.4. Verification of the algorithm. We will now give a mathematical proof of the validity of the algorithm, and define some values of the parameters P, η, R and then K_0 so that for $K \gg K_0$ the algorithm describes a subgame perfect equilibrium of $G(K)$.

$C = \max_{i,a}|g^i(a)|$ is the norm of G, and we put $c = \min \bigcup_i \{|g^i(v) - g^i(w)|, g^i(v), g^i(w)\}$. Let η be fixed by Lemma 3.2 such that, during any punishing sequence, if all punishers are effective the punished player does not get more than $c/4$ as average payoff. Also let δ be a probability of not being an effective punisher such that $I\delta C < c/4$. P_0 is now given by Lemma 3.1 such that every punisher is effective with probability at least $1 - \delta/2$ if he plays repeatedly (i.i.d.) his minimax strategy during $P > P_0$ stages and if the test is computed with the value of η fixed above.

If every punisher is effective with probability at least $1 - \delta$, no player has an incentive to deviate as soon as $C + P(c/4 + I\delta C) < -C + Pc$ (the maximal gain from a deviation is compensated for by the loss of expected payoff during the punishing period). Thus we choose P, $P > P_0$, $P > 4C/c$.

We now determine R, duration of a reward phase, so that any punishing player will have incentives to be effective with at least probability $1 - \delta$ when a deviation occurs.

First, note that during a punishing sequence, a punisher j cares only about the payoffs during the punishing phase and the payoffs during his own reward phase. In fact j's total payoff is not changed if some other punisher k is rewarded or not, if k is older than j then j will die before the reward phase of k, and if k is younger than j the reward phase of k is compensated for during the next compensation phase.

The algorithm does not define explicitly the recommended strategies during a punishing phase. Players just know what will be played after the end of the punishing phase, and "make their own computation" of equilibrium strategies. At such an equilibrium, if R is large enough, any punisher is effective with a probability greater than $1 - \delta$ since any strategy of a punisher j making him an effective punisher with a probability less than $1 - \delta$ is dominated by playing the repeated minimax strategy. Actually by such a change of strategy he may lose at most $2PC$ during the punishing period, but wins at least $(\delta/2)Rc$ of expected reward. Hence we choose R multiple of $2l$, $R > 4PC/\delta c$.

Up to now we have defined values of η, P, and R such that no player has an incentive to deviate from the Main Path before stage $K^i - 2P - R$. It is also clear that i may gain from deviating from the Main Path neither before stage $K^i - P$ (he is then punished), nor after stage $K^i - P + 1$ (he plays a best response in (MP)).

If the overlap lengths are large, a player $j \neq i$ deviating "late" in an overlap will lose more during the Normal Phase of the next overlap when $\tilde{z}^j_-$ is played instead of $\tilde{z}$ that he may gain in the at most $2P + R$ last stages of the overlap of his deviation. To ensure this, a sufficient condition on K^i is that $(K^i - P - 2R)c/2 > (2P + R)2C$. This at last gives a lower bound K_0 such that for $K \gg (K_0, \ldots, K_0)$, the algorithm defines a subgame perfect equilibrium of $G(K)$, with average payoff close to u when K is large.

4. Extensions and variations on Theorem A. In this section we present some theorems similar to Theorem A, and show how their proofs can be adapted from the proof of Theorem A. We concentrate here more on the underlying ideas than on the formal proofs, avoiding heavy notations.

4.1. The discounted OLG $G(K, \Delta)$. In the case where players discount their future payoffs with a discount factor $\Delta \in [0, 1]$ (Δ close to 1 corresponds to "patient" players), we represent by $G^*(K, \Delta)$ and $G(K, \Delta)$ the OLGs constructed from G to K respectively with observable and nonobservable mixed strategies.

Smith (1992) proves two Folk Theorems for discounted OLGs with observable mixed strategies for more than two players. One is called nonuniform since it states that for $u \in V$ and K large being fixed, subgame perfect equilibria of $G^*(K, \Delta)$ leading to payoffs near u exist when Δ is close to 1 (the lower bound Δ_0 on Δ depends on K). The other, called uniform, proves that if the dimension of V is I, any $u \in V$ can be approximated by subgame perfect equilibrium payoffs of $G^*(K, \Delta)$ as soon as K is large and Δ close to 1; these conditions are independent of each other.

With nonobservable mixed strategies, let G, u, and some parameters of the algorithm of §3.3 be fixed. All the inequalities that characterize the subgame perfect equilibrium of our proof are strict and finite in number. Hence they remain true when the future payoffs are little discounted. This shows that Theorem A can be extended to a nonuniform Folk Theorem for discounted OLGs with nonobservable mixed strategies.

4.2. Observable mixed strategies. Repeated games with observable or nonobservable mixed strategies are of two different classes, and generally the set of subgame perfect equilibrium payoffs with observable mixed strategies is not included in the set of subgame perfect equilibrium payoffs with nonobservable mixed strategies. Yet this inclusion seems at least to hold within the limit. In other words when a Folk Theorem is true with nonobservable mixed strategies, it is often also true with observable ones. Indeed algorithms proving Folk Theorems in the nonobservable case are generally easily adaptable to the observable case, since in this last case one can say whether a player used some minimax strategy or not during a punishing sequence. In our case, we get a proof of Theorem 2 from the proof of Theorem A just be substituting in the algorithm at the end of a punishing period P(j) "$r(k) = \alpha_\eta^{j,k}(h_{t_0+1}, \ldots, h_{t_0+P})$," the value of the statistic on the punishing period by "$r(k) = 1$ if player k has played repeatedly m_j^k during the P stages of the punishing period, and $r(k) = 0$ if he has not."

In this new algorithm for OLGs with observable mixed strategies, every punisher has incentives to play repeatedly his minimax strategy against the deviator to get the reward at the end of his life, and knowing this no player has incentives to deviate from the Main Path. In this way we get a quicker proof of Theorem 2 than in Smith (1992).

4.3. Stationary equilibria and periodicity. In our model we focused on stationary subgame perfect equilibrium payoffs, that are streams of payoffs giving the same payoff to every player of the same type (except maybe players of the 0th generation). In fact, the strategies defined by the algorithm have themselves some periodicity properties. First we should emphasize that the equilibrium path is T-periodic. Note also that players' mixed moves depend only on i, the type of the oldest player, on $K^i - t$, the age of the oldest player, $r(j)$, λ, and also on whether a player deviated recently, on the type of this player and on the stage at which he deviated. As a consequence, bounded recall is enough, in the sense that players' moves are a function only of the history during the T last stages. We may then redefine our equilibrium strategies by keeping only T stages of memory; they therefore constitute a *stationary equilibrium.*

4.4. A more general model for OLGs. So far, we only considered OLG's whose structure had some periodicity property. Except for those of the 0th generation, the life of all players lasts the same T stages. Even though it leads to simple notations and eases the understanding of the construction of $G(K)$ and $\mathscr{K}(i)$, perhaps this restriction is not in the original spirit of OLGs. If we drop this periodicity property, we can consider a structure of OLG as a mapping $\Phi = (T, l)$ from $\mathbb{N}$ to $\mathbb{N}^* \times I$ rather than as a vector $K = (K^1, \dots, K^I)$. $\Phi(m) = (T(m), l(m))$ means that after stage $T(m)$ the player of type $l(m)$ is replaced. We assume that T is strictly increasing, and the condition that $l(m)$ takes an infinite number of times every value in I implies that the players are finitely living in the so-constructed OLG $G(\Phi)$. $G(\Phi)$ is of the kind $G(K)$ if and only if $T(m+1) - T(m)$ and $l(m)$ both are I-periodic. In $G(\Phi)$, the durations of overlaps (our new overlap lengths) are $T(m+1) - T(m)$. We claim that subgame perfect equilibrium average payoffs of $G(\Phi)$ may approximate any $u \in V$ if $V \cap \mathbb{R}^I_{++} \neq \varnothing$ when all tend to be large. To see this, note that in the proof of Theorem A, the periodicity of the structure of $G(K)$ is not used, all that is needed is that R, P, and the length of an overlap are large enough. In this way we finally arrive at a Folk Theorem for a broader class of OLGs than the ones with a periodic structure.

4.5. Conclusion. For finitely repeated games, two distinct Nash payoffs in G and a condition on the dimension of V are needed to get a Folk Theorem with observable or nonobservable mixed strategies (see Benoît and Krishna 1985 and Gossner 1995). None of these assumptions are needed to get a Folk Theorem for OLGs. Basically, in both cases one has to be able to reward some player at the end of his life without affecting some other player's total payoff. In finitely repeated games, this explains the need for the *full dimensionality* assumption, that can be dropped in OLGs as a benefit of the fact that players never die simultaneously.

This result completes a series of Folk Theorems, with observable or nonobservable mixed strategies, for infinitely repeated games (with or without discounting), finitely repeated games and overlapping generations games, that hold with a standard information structure. When the information structure is not standard and players get a signal after each turn, several results have been obtained by Lehrer (see, e.g., Lehrer 1991). Clearly, a new series of Folk Theorems is to be developed for such models.

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