Games and Economic Behavior **30,** 44–63 (2000)
doi:10.1006/game.1998.0706, available online at http://www.idealibrary.com on IDEAL®

# Comparison of Information Structures

Olivier Gossner[1]

*THEMA, UMR 7536, Université Paris Nanterre, 200 Avenue de la République, 92001 Nanterre CEDEX, France and CORE, 34 voie du Roman Pays, B-1348 Louvain-la-Neuve, Belgium*
E-mail: Olivier.Gossner@u-paris10.fr

Received November 26, 1996

We introduce the notion of an information structure $\mathcal{I}$ as being richer than another $\mathcal{J}$ when for every game $G$, all correlated equilibrium distributions of $G$ induced by $\mathcal{J}$ are also induced by $\mathcal{I}$. In particular, if $\mathcal{I}$ is richer than $\mathcal{J}$ then $\mathcal{I}$ can make all agents as well off as $\mathcal{J}$ in any game. We also define $\mathcal{J}$ to be faithfully reproducible from $\mathcal{I}$ when all the players can compute from their information in $\mathcal{I}$ "new information" that reproduces what they could have received from $\mathcal{J}$. Our main result is that $\mathcal{I}$ is richer than $\mathcal{J}$ if and only if $\mathcal{J}$ is faithfully reproducible from $\mathcal{I}$. *Journal of Economic Literature* Classification Number: C72. 

*Key Words:* Information structure; correlated equilibrium; statistical experiment; value of information.

## 1. INTRODUCTION

For one agent, Blackwell's *comparison of statistical experiments* provides a general theory for the value of information (1951, 1953). A statistical experiment is more informative than another one when it brings a better payoff to the statistician in every decision problem. Blackwell's theorem asserts that a statistical experiment is more informative than another one if and only if the statistician can reproduce the information from the former to the latter.

Since then, many attempts have been done to generalize this result to multi-agent situations. Unfortunately, it has been observed by several authors that more information is not always profitable in interactive contexts. For instance, Hirshleifer (1971) observed that public disclosure of

[1] The author is grateful to Bernard de Meyer, Jean-Francois Mertens, Sylvain Sorin, two anonymous referees, and an associate editor for enriching comments. Part of this work was done at the Center for Interactive Decision Theory of Jerusalem and at Universitat Pompeu Fabra of Barcelona.



0899-8256/00 $35.00

information can make all agents worse off by ruling out opportunities to insure.

Akerlof's (1970) *market for lemons* gives another example of a negative value of information. In this model no trade of a used car is possible if the value of the car is known to the seller but not to the buyer. Would both agents or none of them know the car's value, trade would be possible and would benefit to both the buyer and the seller.

Green and Stokey (1981) consider a principal-agent model in which the principal receives some information on the state of nature and sends a signal to the agent, after which the agent takes a decision that affects both players. Again, their observation is that better information generally does not improve welfare.

Bassan, Scarsini, and Zamir, (1997) finally exhibit two-players games with incomplete information showing that "*almost every situation is conceivable: Information can be beneficial for all players, just for the one who does receive it, or, less intuitively, just for the one who does not receive it, or it could be bad for both*."

The fact that information can hurt the agent who receives it may be counter-intuitive at first sight. Neyman (1991) clarifies the fact that more information is always valuable to an agent as long as the others are not aware of it. Otherwise, the other agents may behave in a way that hurts the informed one.

Information structures (Aumann, 1974, 1987) are the natural extension of statistical experiments to multi-agent setups. An information structure describes all player's information about the state of nature as well as higher order beliefs such as the information each player has on other player's information on the state of nature, and so on.... We shall represent information structures by probabilities over a set of payoff relevant states of nature times a product space of sets of signals for each player. Given a strategic game $G$ and an information structure $\mathcal{I}$, the game $G$ extended by $\mathcal{I}$ is the game in which players first receive information according to $\mathcal{I}$, and second play in $G$. The distributions on the actions in $G$ induced by Nash equilibria of this extended game are called the correlated equilibrium distributions of $G$ induced by $\mathcal{I}$.

Following a similar method to Blackwell, we present two ways of comparing information structures, and prove their equivalence.

First we define $\mathcal{I}$ to be richer than $\mathcal{J}$ when for any game $G$, all correlated equilibrium distributions of $G$ induced by $\mathcal{J}$ are also correlated equilibrium distributions of $G$ induced by $\mathcal{I}$. In particular, $\mathcal{I}$ richer than $\mathcal{J}$ can always make all agents as well off than $\mathcal{J}$.

Second we define an interpretation $\phi$ from $\mathcal{I}$ to $\mathcal{J}$ as a way for players to compute from their signals in $\mathcal{J}$ some interpreted signals that they could have received in $\mathcal{I}$. We call an interpretation $\phi$ a compatible interpretation when the probability distribution it induces on the product space of states of nature and of interpreted signals is equal to the probability distribution given by $\mathcal{J}$. Moreover, a compatible interpretation is faithful when every player has the same information on the state of nature and on the interpreted signals of the other players, given his original signal (given by $\mathcal{I}$) or given his interpreted one. In other words, $\phi$ is faithful if no player loses information by computing his interpreted signal and forgetting his original one. This condition implies that if $f$ is a Nash equilibrium of $G$ extended by $\mathcal{I}$ and if $\phi$ is a faithful interpretation from $\mathcal{I}$ to $\mathcal{J}$, it is a Nash equilibrium of $G$ extended by $\mathcal{J}$ to reproduce signals in $\mathcal{J}$ from the signals in $\mathcal{I}$ according to $\phi$, then to follow $f$.

The relations "$\mathcal{I}$ is richer than $\mathcal{J}$" and "there exists a faithful interpretation from $\mathcal{I}$ to $\mathcal{J}$" are reflexive and transitive. They both permit comparison of information structures. In this article we prove the equivalence between these two relations. Namely, $\mathcal{I}$ is richer than $\mathcal{J}$ if and only if there exists a faithful interpretation from $\mathcal{I}$ to $\mathcal{J}$.

The approach we follow here is to compare information structures according to the correlated equilibrium distributions they induce in every game, then to characterize this relation in terms involving both information structures and no external game. This "dual" approach is similar to the methodology followed by Monderer and Samet, (1996) who study the proximity of information structures.

The revelation principle (Myerson, 1982) makes correlated equilibrium distributions easy to implement since it shows that any correlated equilibrium distribution $\mu$ of a game $G$ is induced by the information structure $\mathcal{I}(\mu)$ with signals being the actions in $G$ and with probability $\mu$ on the signals. A consequence of our definition is that if $\mu$ is a correlated equilibrium distribution of $G$ and if $\mathcal{I}$ is richer than $\mathcal{I}(\mu)$, $\mu$ is also induced by $\mathcal{I}$. Hence we provide a whole class of information structures that allow to implement a given correlated equilibrium distribution.

This type of reasoning has already proved useful in repeated games with signals. Lehrer (1990) exhibits equilibria of repeated games with signals where players first generate an information structure $\mathcal{I}$ which is richer than $\mathcal{J}$, then play as if signals in $\mathcal{J}$ had been sent by an independent correlation device. In a more general setup, we show in Gossner (1998) that if an information structure $\mathcal{I}$ can be generated through communication and if $\mathcal{I}$ is richer than $\mathcal{J}$, then $\mathcal{J}$ can be generated as well.

After preliminaries in Sec. 2, we present the main result and some examples in Sec. 3. Section 4 is devoted to the proof of the main result and to a corollary. In general, we cannot assume the interpretations to be deterministic, players may randomize to compute their new signals. In Sec. 5 we exhibit conditions under which the interpretations can be assumed to be deterministic. We also study the equivalence classes for the relation "$\mathcal{I}$

is richer than $\mathcal{J}$ and $\mathcal{J}$ is richer than $\mathcal{I}$." Our main result is first stated for games with complete information, Sec. 6 includes the extension to incomplete information. Finally we examine more closely what the exact connections between our work and Blackwell's comparison of experiments in Sec. 7 are, and conclude in Sec. 8.

## 2. PRELIMINARIES

### 2.1. *General Notations*

$I = \{1, \ldots, I\}$ is a finite set of players. Given a collection $(Z^i)_{i \in I}$ of sets, $Z$ represents $\prod_i Z^i$, and $Z^{-i}$ is $\prod_{j \neq i} Z^j$. Similarly, $z^i$ and $z^{-i}$ are the canonical projections of $z \in Z$ on $Z^i$ and $Z^{-i}$. Given a topological set $W$, $\Delta(W)$ denotes the set of regular probability measures over the Borel $\sigma$-algebra on $W$. If $P$ is a probability measure, $\mathbf{E}_P$ represents the expectation operator over $P$. For $P \in \Delta(Z)$ with $Z = \Pi_i Z^i$, $P(z^i)$ and $P(z^{-i})$ stand for $P(\{z^i\} \times Z^{-i})$ and $P(\{z^{-i}\} \times Z^i)$.

### 2.2. *Games Extended by Information Structures*

A **compact game** $G = ((S^i)_i, g)$ is given by a compact set of strategies $S^i$ for each player $i$ and by a continuous payoff function $g$ from $S$ to $\mathbb{R}^I$. The set of mixed strategies for player $i$ is $\Sigma^i = \Delta(S^i)$, and $g$ is extended to $\Sigma$ by $g(\sigma) = \mathbf{E}_\sigma g(s)$ (the product set $\Pi_i \Delta(S^i)$ is identified to be a subset of $\Delta(S)$).

An **information structure** $\mathcal{I} = ((X^i)_i, \mu)$ is defined by a family of finite sets of signals $X^i$ and by a probability measure $\mu$ over $X$. When $x$ is drawn according to $\mu$, $i$ is informed about his signal $x^i$.

DEFINITION 1. Given a compact game $G$ and an information structure $\mathcal{I}$, $\Gamma(\mathcal{I}, G)$ represents the game $G$ **extended by** $\mathcal{I}$ in which:

- $x \in X$ is drawn according to $\mu$, each player $i$ is informed about $x^i$;
- each player $i$ chooses $\sigma^i \in \Sigma^i$;
- the payoffs are given by $g(\sigma)$.

A strategy for player $i$ is a mapping $f^i$ from $X^i$ to $\Sigma^i$, and the payoff function of $\Gamma(\mathcal{I}, G)$ is given by $g_{\mathcal{I}}(f) = \mathbf{E}_\mu g(f(x))$.

$D(\mathcal{I}, G)$ denotes the set of correlated equilibrium distributions of $G$ induced by $\mathcal{I}$. It is the set of distributions on $S$ that are images of $\mu$ by Nash equilibria of $\Gamma(\mathcal{I}, G)$.

For $x^i \in X^i$ such that $\mu(x^i) > 0$, $p(x^i) \in \Delta(X^{-i})$ denotes the conditional probability of $\mu$ given $x^i$ over $X^{-i}$:

$$p(x^i)(x^{-i}) = \mu(x^{-i}|x^i) = \frac{\mu(x^{-i}, x^i)}{\mu(x^i)}.$$

Recall that $\mu(x^i)$ stands for $\mu(x^i \times X^{-i})$.

*Remark* 1. $f$ is a Nash equilibrium of $\Gamma(\mathcal{I}, G)$ if and only if for every player $i$

$$f^i(x^i) \in \operatorname{Arg}\max_{\tau^i \in \Sigma^i} \mathbf{E}_{p(x^i)} g^i(\tau^i, f^{-i}(x^{-i})), \qquad \mu\text{-a.s.}$$

This is simply a consequence of the relation

$$g^i_{\mathcal{I}}(f) = \sum_{x^i \in X^i} \mu(x^i) \mathbf{E}_{p(x^i)} g^i(f^i(x^i), f^{-i}(x^{-i})).$$

This characterization expresses the well known fact that at an equilibrium, each player maximizes his expected payoff conditional to his information.

## 3. COMPARISON OF INFORMATION STRUCTURES

In this section, we introduce two ways of comparing two information structures $\mathcal{I} = ((X^i)_i, \mu)$ and $\mathcal{J} = ((Y^i)_i, \nu)$.

The first definition says that $\mathcal{I}$ is richer than $\mathcal{J}$ whenever $\mathcal{I}$ induces all the correlated equilibrium distributions that are induced by $\mathcal{J}$.

DEFINITION 2. $\mathcal{I}$ **is richer than** $\mathcal{J}$ when for every compact game $G$

$$D(\mathcal{I}, G) \supseteq D(\mathcal{J}, G).$$

For the second definition, we imagine that players receive signals from $\mathcal{I}$, and define conditions under which they can reproduce signals that could have been issued by $\mathcal{J}$. An **interpretation mapping for player** $i$ from $\mathcal{I}$ to $\mathcal{J}$ is an application $\phi^i$ from $X^i$ to $\Delta(Y^i)$. When $x^i$ is $i$'s signal in $\mathcal{I}$, the interpreted signal in $\mathcal{J}$ is $y^i$ with probability $\phi^i(x^i)(y^i)$. An **interpretation** from $\mathcal{I}$ to $\mathcal{J}$ is a family $\phi = (\phi^i)_i$ of interpretation mappings for all the players. $\phi$ and $\phi^{-i} = (\phi^j)_{j \neq i}$ define mappings from $X$ to $\Delta(Y)$ and from $X^{-i}$ to $\Delta(Y^{-i})$ when $\Pi_i \Delta(Y^i)$ and $\Pi_{j \neq i} \Delta(Y^i)$ are identified to be subsets of $\Delta(Y)$ and of $\Delta(Y^{-i})$.

DEFINITION 3. A **compatible interpretation** from $\mathcal{I}$ to $\mathcal{J}$ is an interpretation $\phi$ from $\mathcal{I}$ to $\mathcal{J}$ such that the image of $\mu$ by $\phi$ is $\nu$, i.e., such that for every $y \in Y$, $\mathbf{E}_\mu \phi(x)(y) = \nu(y)$.

For the remainder of the section, $\phi$ represents a compatible interpretation from $\mathcal{I}$ to $\mathcal{J}$. $P_\phi$ denotes the probability induced on $X \times Y$ by $\mu$ and the transition probability $\phi$ (explicitly $P_\phi(x, y) = \mu(x)\phi(x)(y)$). The marginals of $P_\phi$ on $X$ and $Y$ are $\mu$ and $\nu$, respectively.

We shall say that $\phi$ is faithful whenever no player loses information about the interpreted signal of the others by relying on his interpreted signal and forgetting his original one.

The probabilities on $Y^{-i}$ defined by $r(x^i)(y^{-i}) = P_\phi(y^{-i}|x^i)$ and $q(y^i)(y^{-i}) = P_\phi(y^{-i}|y^i)$ for $P_\phi(x^i, y^i) > 0$ represent the conditional probabilities "before interpretation" and "after interpretation" over the interpreted signals of players others than $i$ when $i$'s signal is $x^i$ and $i$'s interpreted signal is $y^i$. We shall view $q(y^i)$ and $r(x^i)$ as random vectors with values in $\Delta(Y^{-i})$. Note that $P_\phi(y^i|x^i) = \phi^i(x^i)(y^i)$ and that $r(x^i) = \mathbf{E}_{p(x^i)}\phi^{-i}(x^{-i})$.

Definition 4. An interpretation $\phi$ from $\mathcal{I}$ to $\mathcal{J}$ is **faithful** if it is compatible and if for every $i$, $q(y^i) = r(x^i)$ $P_\phi$-a.s. If there exists a faithful interpretation from $\mathcal{I}$ to $\mathcal{J}$, we say that $\mathcal{J}$ is **faithfully reproducible** from $\mathcal{I}$.

Intuitively, $\phi$ is faithful when $y^i$ is a sufficient statistic for $y^{-i}$ for player $i$. This is stated more explicitly in Sec. 7.2 where we provide equivalent definitions of a faithful interpretation using Blackwell's comparison of statistical experiments.

Our main result asserts the equivalence between the two comparisons of information structures. Namely:

Theorem 1 (Main Theorem). *$\mathcal{I}$ is richer than $\mathcal{J}$ if and only if $\mathcal{J}$ is faithfully reproducible from $\mathcal{I}$.*

Example 1. We represent two players' information structures by matrices. Each cell contains its probability to be drawn, player 1 is informed about the row, and player 2 about the column.

Consider the information structures $\mathcal{I}_1$ and $\mathcal{I}_2$:

| | $a_1$ | $b_1$ |
|---|---|---|
| $x_1$ | 1/3 | 1/3 |
| $y_1$ | 1/3 | 0 |

$\mathcal{I}_1$

| | $a_2$ | $a_2'$ | $b_2$ |
|---|---|---|---|
| $x_2$ | 1/6 | 0 | 1/6 |
| $x_2'$ | 0 | 1/6 | 1/6 |
| $y_2$ | 1/6 | 1/6 | 0 |

$\mathcal{I}_2$

and the classical game $G$ of “Chicken” together with one of its correlated equilibrium distributions $D$:

| | $L$ | $R$ |
|---|---|---|
| $T$ | 6,6 | 2,7 |
| $B$ | 7,2 | 0,0 |

$G$

| | $L$ | $R$ |
|---|---|---|
| $T$ | 1/3 | 1/3 |
| $B$ | 1/3 | 0 |

$D$

It follows from the revelation principle that the strategies $(f_1^1, f_1^2)$ of $\Gamma(\mathcal{I}_1, G)$ defined by

$$\begin{cases} f_1^1(x_1) = T \\ f_1^1(y_1) = B \end{cases} \qquad \begin{cases} f_1^2(a_1) = L \\ f_1^2(b_1) = R \end{cases}$$

induce $D$ as correlated equilibrium distribution of $G$.

$D$ is also a correlated equilibrium distribution of $G$ induced by the strategies $(f_2^1, f_2^2)$

$$\begin{cases} f_2^1(x_2) = T \\ f_2^1(x_2') = T \\ f_2^1(y_2) = B \end{cases} \qquad \begin{cases} f_2^2(a_2) = L \\ f_2^1(a_2') = L \\ f_2^1(b_2) = R \end{cases}$$

of $\Gamma(\mathcal{I}_2, G)$.

Now let $\phi = (\phi^1, \phi^2)$ be the interpretation from $\mathcal{I}_2$ to $\mathcal{I}_1$ defined by

$$\begin{cases} \phi^1(x_2) = x_1 \\ \phi^1(x_2') = x_1 \\ \phi^1(y_2) = y_1 \end{cases} \qquad \begin{cases} \phi^2(a_2) = a_1 \\ \phi^2(a_2') = a_1 \\ \phi^2(b_2) = b_1 \ . \end{cases}$$

With $\phi^1$, player 1 “identifies” signals $x_2$ and $x_2'$ of $\mathcal{I}_2$ into $x_1$, $y_2$ is renamed $y_1$. At the same time, player 2 “identifies” $a_2$ and $a_2'$ into $a_1$, and $b_2$ is renamed $b_1$. One verifies easely that $\phi$ is a compatible interpretation. To check that $\phi$ is faithful, we compute for instance the conditional probabilities $p_2(x_2) = 1/2a_2 + 1/2b_2$, $r(x_2) = \mathbf{E}_{p_2(x_2)}\phi^2 = 1/2a_1 + 1/2b_1 = p_1(x_1)$.

For an example of a compatible interpretation which is not faithful, consider the information structure $\mathcal{I}_3$:

| | $a_3$ | $b_3$ |
|---|---|---|
| $x_3$ | 0 | 1/3 |
| $x_3'$ | 1/3 | 0 |
| $y_3$ | 1/3 | 0 |

$\mathcal{I}_3$

and the interpretation from $\mathcal{I}_3$ to $\mathcal{I}_1$ defined by $\phi'^1(x_3) = \phi'^1(x'_3) = x_1$, $\phi'^1(y_3) = y_1$ and $\phi'^2(a_3) = a_1$, $\phi'^2(b_3) = b_1$. In fact, $\phi'$ is compatible, but one sees that $r'(x_3) = b_1 \neq p_1(x_1)$.

Note also that $D$ is not a correlated equilibrium distribution of $G$ induced by $\mathcal{I}_3$. In fact, one sees that $\mathcal{I}_3$ is equivalent to a public correlation device, so every correlated distribution induced by $\mathcal{I}_3$ in any game must be a convex combination of Nash equilibria.

## 4. PROOF OF THE MAIN RESULT—COROLLARY

This section is devoted to a proof of Theorem 1 and to a corollary. $\mathcal{I} = ((X^i)_i, \mu)$ and $\mathcal{J} = ((Y^i)_i, \nu)$ are fixed, as well as $p(x^i)(x^{-i}) = \mu(x^{-i}|x^i)$ and $q(y^i)(y^{-i}) = \nu(y^{-i}|y^i)$. When a compatible interpretation $\phi$ from $\mathcal{I}$ to $\mathcal{J}$ is known, $P_\phi$ is the probability induced by $\mu$ and $\phi$ on $X \times Y$, and $r(x^i)(y^{-i}) = P_\phi(y^{-i}|x^i)$.

### 4.1. *Construction of Strategies in* $\Gamma(\mathcal{I}, G)$ *from* $\phi$ *and Strategies in* $\Gamma(\mathcal{J}, G)$

Assume that $\phi$ is an interpretation from $\mathcal{I}$ to $\mathcal{J}$. Let $G$ be a compact game, and $f$ a $I$-tuple of strategies in $\Gamma(\mathcal{J}, G)$. A $I$-tuple of strategies $e$ in $\Gamma(\mathcal{I}, G)$ is defined by $e^i(x^i) = \mathbf{E}_{\phi^i(x^i)} f^i(y^i)$ (for every Borel subset $B^i$ of $S^i$, $e^i(x^i)(B^i) = \mathbf{E}_{\phi^i(x^i)} f^i(y^i)(B^i)$). $e^i$ is the strategy that corresponds to:

- pick $y^i$ according to $\phi^i(x^i)$ when $x^i$ is the signal received from $\mathcal{I}$;
- play $f^i(y^i)$ in $G$ (as if $y^i$ had been received from $\mathcal{J}$).

LEMMA 2. *If $\phi$ is a compatible interpretation, $e$ and $f$ induce the same distribution on $S$.*

*Proof.* Let $I_e$ and $I_f$ be the image distributions of $\mu$ and $\nu$ by $e$ and $f$ on $S$. $e$ and $f$ define mappings from $X$ and $Y$ to $\Delta(S)$, and for every product $B = B^1 \times \ldots \times B^I$ of Borel subsets of $S^1, \ldots, S^I$:

$$I_e(B) = \mathbf{E}_\mu e(x)(B) = \mathbf{E}_\mu \mathbf{E}_{\phi(x)} f(y)(B) = \mathbf{E}_\nu f(y)(B) = I_f(B)$$

■

LEMMA 3. *If $\phi$ is faithful and if $f$ is a Nash equilibrium of $\Gamma(\mathcal{J}, G)$, $e$ is a Nash equilibrium of $\Gamma(\mathcal{I}, G)$.*

*Proof.* We use Remark 1. For $x^i \in X^i$ such that $\mu(x^i) > 0$ and $\sigma^i \in \Sigma^i$, we get by successively using the definition of $e$, the definition of $r$, and the fact that $\phi$ is faithful

$$\begin{aligned}\mathbf{E}_{p(x^i)}g^i(\sigma^i, e^{-i}(x^{-i})) &= \mathbf{E}_{p(x^i)}\mathbf{E}_{\phi^{-i}(x^{-i})}g^i(\sigma^i, f^{-i}(y^{-i}))\\ &= \mathbf{E}_{r(x^i)}g^i(\sigma^i, f^{-i}(y^{-i}))\\ &= \mathbf{E}_{\phi^i(x^i)}\mathbf{E}_{q(y^i)}g^i(\sigma^i, f^{-i}(y^{-i})).\end{aligned}$$

Since $f$ is a Nash equilibrium of $\Gamma(\mathcal{J}, G)$, this is at most

$$\begin{aligned}\mathbf{E}_{\phi^i(x^i)}\mathbf{E}_{q(y^i)}g^i(f^i(y^i), f^{-i}(y^{-i})) &= \mathbf{E}_{r(x^i)}g^i(e^i(x^i), f^{-i}(y^{-i}))\\ &= \mathbf{E}_{p(x^i)}g^i(e^i(x^i), e^{-i}(x^{-i})).\end{aligned}$$

■

This completes the first part of the proof of Theorem 1.

### 4.2. *Construction of a Faithful Interpretation if $\mathcal{I}$ is Richer Than $\mathcal{J}$*

#### 4.2.1. *Sketch of the Proof*

It is easy to prove the existence of a compatible interpretation from $\mathcal{I}$ to $\mathcal{J}$ when $\mathcal{I}$ is richer than $\mathcal{J}$. To do this, consider the game $G$ whose spaces of strategies are $S^i = Y^i$, and with payoff function $g \equiv 0$. The strategies of $\Gamma(\mathcal{J}, G)$ defined by $f^i(y^i) = y^i$ form a Nash equilibrium, and the induced distribution on the actions of $G$ is $\nu$. Consider a Nash equilibrium $e$ of $\Gamma(\mathcal{I}, G)$ inducing the same distribution on $S$. If we set $\phi^i(x^i) = e^i(x^i)$, $\phi$ defines a compatible interpretation from $\mathcal{I}$ to $\mathcal{J}$.

To construct a faithful interpretation from $\mathcal{I}$ to $\mathcal{J}$ is slightly more complicated. We do it by constructing a game $G$ and a Nash equilibrium of $\Gamma(\mathcal{J}, G)$ in which each player reveals his signal and his conditional probability over the signals of the others. This game $G$ will not be compact, so we start by assuming that the inclusion $D(\mathcal{I}, G) \supseteq D(\mathcal{J}, G)$ is also satisfied when $G$ is an upper semicontinuous game. We prove the existence of a faithful interpretation under this assumption, then we complete the proof of the main theorem using approximations of upper semicontinuous games by compact games.

#### 4.2.2. *Case Where the Payoff Function May Be Upper Semicontinuous*

An **upper semicontinuous (or usc) game** is given by $((S^i)_i, g)$, where the sets $(S^i)_i$ are compact, and where $g : S \to (\mathbb{R} \cup \{-\infty\})^I$ is an upper semicontinuous payoff function. For $G$ an usc game and $\mathcal{I}$ an information structure, $\Gamma(\mathcal{I}, G)$ and $D(\mathcal{I}, G)$ are defined as in the case of a compact game.

$\mathcal{J}$ being fixed, we construct an usc game $G$ from $\mathcal{J}$ as follows: an element of $S^i = Y^i \times \Delta(Y^{-i})$ is a couple $(y^i, \delta^i)$, the payoff of player $i$ is $g^i(y, \delta) =$

$\ln \delta^i(y^{-i})$ if $\delta^i(y^{-i}) > 0$, $g^i(y, \delta) = -\infty$ otherwise. The payoff of $i$ does not depend on $\delta^{-i}$ nor on $y^i$, we write it $h^i(y^{-i}, \delta^i)$. For notational convenience, $h^i(\gamma, \delta^i) = \mathbf{E}_\gamma h^i(y^{-i}, \delta^i)$ if $\gamma \in \Delta(Y^{-i})$.

In $G$, each player announces a signal and a probability over the signals of the others. The payoff function of $G$ is designed such that in an extended game $\Gamma(\mathcal{I}, G)$, each player has incentives to announce as probability his conditional probability over the signals announced by the others. More precisely, consider an $I$-tuple $e$ of strategies in $\Gamma(\mathcal{I}, G)$. For $x^i \in X^i$, $e^i(x^i)$ is a probability measure over $Y^i \times \Delta(Y^{-i})$. We denote by $e^i_Y(x^i)$ and $e^i_\Delta(x^i)$ its marginals on $Y^i$ and $\Delta(Y^{-i})$, respectively. $e$ induces with $\mu$ a probability $P_e$ on $X \times Y \times \Pi_i \Delta(Y^{-i})$. Let $\gamma^i(x^i) = P_e(y^{-i}|x^i)$ be the conditional probability on $x^i$ of the signal $y^{-i}$ announced by the other players.

LEMMA 4. *$e$ is a Nash equilibrium of $\Gamma(\mathcal{I}, G)$ if and only if for all $i$, $e^i_\Delta(x^i)$ is the Dirac mass at $\gamma(x^i)$ $\quad \mu$-a.s.*

*Proof.* From Remark 1, $e$ is a Nash equilibrium if and only if for all $i$:

$$e^i_\Delta(x^i) \in \arg \max_{\sigma^i_\Delta \in \Delta(\Delta(Y^{-i}))} \mathbf{E}_{\sigma^i_\Delta} h^i(\gamma^i(x^i), \delta^i) \qquad \mu\text{-a.s.}$$

See that for $\gamma, \delta^i \in \Delta(Y^{-i})$

$$h^i(\gamma, \gamma) - h^i(\gamma, \delta^i) = \sum_{y^{-i} \in Y^{-i}} \gamma(y^{-i}) \ln \frac{\gamma(y^{-i})}{\delta^i(y^{-i})} = D(\gamma \,\|\, \delta^i),$$

where $D(\gamma \,\|\, \delta^i)$ is the relative entropy (or Kullback Leibler distance) between $\gamma$ and $\delta^i$ (see for instance Cover and Thomas, (1991)). By property of the relative entropy, $D(\gamma \,\|\, \delta^i) \geq 0$, with $D(\gamma \,\|\, \delta^i) = 0$, if and only if $\gamma = \delta^i$. Therefore

$$\mathbf{E}_{\sigma^i_\Delta} h^i(\gamma(x^i), \delta^i) \leq h^i(\gamma(x^i), \gamma(x^i))$$

with equality only for $\sigma^i_\Delta$ Dirac mass at $\gamma(x^i)$. ■

PROPOSITION 5. *Assume that $D(\mathcal{I}, G) \supseteq D(\mathcal{J}, G)$ for the previously defined usc game $G$, then $\mathcal{J}$ is faithfully reproducible from $\mathcal{I}$.*

*Proof.* Consider the strategies in $\Gamma(\mathcal{J}, G)$ defined by $f^i(y^i) = (y^i, q(y^i))$. By Lemma 4, $f$ is a Nash equilibrium. Let $e$ be a Nash equilibrium inducing the same distribution on the actions of $G$. An interpretation $\phi$ is given by $\phi^i(x^i) = e^i_Y(x^i)$. The proposition is a consequence of the next two lemmas.

LEMMA 6. *$\phi$ is a compatible interpretation from $\mathcal{I}$ to $\mathcal{J}$.*

*Proof.* The marginals on $Y$ of the distributions induced by $e$ and $f$ on $S$ are the image of $\mu$ by $\phi$ and $\nu$, respectively. ■

LEMMA 7. *$\phi$ is faithful.*

*Proof.* Note that the probability $P_\phi$ induced by $\mu$ and $\phi$ on $X \times Y$ is the marginal of $P_e$ on $X \times Y$. Therefore $r(x^i)(y^{-i}) = P_\phi(y^{-i}|x^i) = P_e(y^{-i}|x^i) = \gamma(x^i)(y^{-i})$. Take $x^i, y^i$ such that $P_\phi(x^i, y^i) > 0$. Since $e$ is a Nash equilibrium, $e^i_\Delta(x^i)$ is the Dirac mass at $\gamma(x^i)$. Then $e^i(x^i)(y^i, \gamma(x^i)) > 0$. Because $e$ and $f$ induce the same distribution on the actions of $G$, there exists $y'^i \in Y^i$ such that $f^i(y'^i)(y^i, \gamma(x^i)) > 0$. By definition of $f$ we have $y'^i = y^i$, and $q(y^i) = \gamma(x^i) = r(x^i)$. ■

4.2.3. *Case where the payoff function is continuous*

Here we approximate the previously defined game $G$ by a family of compact games $G_K$. We study the best response correspondence of $G_K$, then we construct an interpretation from $\mathcal{I}$ to $\mathcal{J}$ from a Nash equilibrium of $\Gamma(\mathcal{J}, G)$ and prove that it is close to a faithful interpretation.

First, we need to define an **$\varepsilon$-faithful** interpretation for $\varepsilon > 0$. On a finite set $Z$, we shall use the metric on $\Delta(Z)$ given by $d(\rho_1, \rho_2) = \max_{z \in Z} |\rho_1(z) - \rho_2(z)|$.

DEFINITION 5. For $\varepsilon > 0$, a compatible interpretation $\phi$ is an $\varepsilon$-faithful interpretation when for all $i$, $d(r(x^i), q(y^i)) \leq \varepsilon$ $P_\phi$-a. s. .

PROPOSITION 8. *$\mathcal{J}$ is faithfully reproducible from $\mathcal{I}$ if and only if there exists an $\varepsilon$-faithful interpretation for all $\varepsilon > 0$.*

*Proof.* The direct proof is obvious since a faithful interpretation is also an $\varepsilon$-faithful interpretation. For all $\varepsilon > 0$, the set of $\varepsilon$-faithful interpretations from $\mathcal{I}$ to $\mathcal{J}$ is compact in the set of interpretations from $\mathcal{I}$ to $\mathcal{J}$ endowed with the topology associated with the metric $D(\phi_1, \phi_2) = \max_x d(\phi_1(x), \phi_2(x))$. If these sets are non-empty, their intersection is also non-empty, hence it contains a faithful interpretation. ■

For $K < 0$, let $G_K$ be the compact game whose spaces of strategies are $S^i$, and with payoff function $g^i_K(y, \delta) = \max\{g^i(y, \delta), K\}$. Again $h^i_K(y^{-i}, \delta^i)$ stands for $g^i_K(y, \delta)$, and $h^i_K(\gamma, \delta^i) = \mathbf{E}_\gamma h^i_K(y^{-i}, \delta^i)$ if $\gamma \in \Delta(Y^{-i})$. $G_K$ is thus defined as $G$ except that all payoffs lower than $K$ are set to $K$. The next lemma characterizes the best response correspondence of $G_K$.

LEMMA 9. *Let $\gamma \in \Delta(Y^{-i})$, and $\beta \in \arg\max_{\beta' \in \Delta(Y^{-i})} h^i_K(\gamma, \beta')$. There exists a subset $J$ of $Y^{-i}$ such that $\beta(z) = \gamma(z)/\sum_{z' \in J} \gamma(z')$ if $z \in J$, $\beta(z) = 0$ if $z \notin J$, and $z \in J$ if $\gamma(z) > -1/K$.*

*Proof.* Take $\beta \in \arg\max_{\beta' \in \Delta(Y^{-i})} h^i_K(\gamma, \beta')$, and $\beta' \in \Delta(Y^{-i})$. Let $p_m = \exp(K)$. One has

$$h^i_K(\gamma, \beta') = \sum_{\beta'(z) < p_m} \gamma(z)K + \sum_{\beta'(z) \geq p_m} \gamma(z) \ln \beta'(z).$$

Then if $\beta(z) < p_m$, $\beta(z) = 0$. Maximizing $\sum_{\beta'(z)\geq p_m} \gamma(z)\ln\beta'(z)$ shows that if $\beta(z) \neq 0$,

$$\beta(z) = \frac{\gamma(z)}{\sum_{\beta(z')\geq p_m} \gamma(z')}.$$

Let $J = \{z, \beta(z) > 0\}$, we have to prove that $z \in J$ if $\gamma(z) > -1/K$. Take $z_0 \notin J$, and let $J_0 = J \cup \{z_0\}$. Define $\beta_0$ by $\beta_0(z) = \gamma(z)/\sum_{z'\in J_0}\gamma(z')$ if $z \in J_0$, and $\beta_0(z) = 0$ otherwise. We have

$$h_K^i(\gamma, \beta_0) \geq \sum_{z\notin J_0} \gamma^i(z)K + \sum_{z\in J} \gamma(z)\ln\frac{\gamma(z)}{\sum_{z'\in J_0}\gamma(z')}.$$

Then with $a = \sum_{z\in J}\gamma(z)$, and $b = \gamma(z_0)$

$$\begin{aligned} h_K^i(\gamma, \beta_0) - h_K^i(\gamma, \beta) &\geq b\ln b - Kb + a\ln a - (a+b)\ln(a+b) \\ &\geq b\ln b - Kb + (1-b)\ln(1-b) \quad \text{since } a+b \leq 1 \\ &\geq \ln\frac{1}{2} - Kb. \end{aligned}$$

Therefore $h_K^i(\gamma, \beta_0) > h_K^i(\gamma, \beta)$ if $\gamma(z_0) > -1/K$, so $z_0 \in J$ if $\gamma(z_0) > -1/K$. ■

The next lemma shows that the best response correspondences of $G_K$ and of $G$ get uniformly close as $K$ tends to $-\infty$.

LEMMA 10. *For $\varepsilon > 0$, there exists $K < 0$ such that for all $\gamma \in \Delta(Y^{-i})$, $\beta \in \arg\max_{\beta'\in\Delta(Y^{-i})} h_K^i(\gamma, \beta')$ implies $d(\gamma, \beta) < \varepsilon$.*

*Proof.* Take $0 < \varepsilon < 1/2$. As $Y^{-i}$ is finite, we can choose $K \leq -1/\varepsilon$ such that for all $\gamma \in \Delta(Y^{-i})$, $\sum_{z\in J}\gamma(z) > 1 - \varepsilon/2$ with $J = \{z \in Y^{-i}, \gamma(z) > -1/K\}$. Take $\beta \in \arg\max_{\beta'} h_K^i(\gamma, \beta')$. If $\beta(z) = 0$, $\gamma(z) \leq -1/K \leq \varepsilon$. If $\beta(z) \neq 0$

$$|\gamma(z) - \beta(z)| = \gamma(z)\left(\frac{1}{\sum_{z'\in J}\gamma(z')} - 1\right) \leq \frac{1}{1-\frac{\varepsilon}{2}} - 1 \leq \varepsilon$$

■

Now we can construct $\varepsilon$-faithful interpretations.

PROPOSITION 11. *If $\mathcal{I}$ is richer than $\mathcal{J}$, there exists an $\varepsilon$-faithful interpretation from $\mathcal{I}$ to $\mathcal{J}$ for all $\varepsilon > 0$.*

*Proof.* For $\varepsilon > 0$, let $K$ be chosen such that $\beta \in \arg\max_{\beta'} h^i_K(\gamma, \beta')$ implies $d(\gamma, \beta) < \varepsilon/2$. To $y^i \in Y^i$ we associate

$$f^i_\Delta(y^i) \in \arg \max_{\beta \in \Delta(Y^{-i})} h^i_K(q(y^i), \beta).$$

The $I$-tuple of strategies $f$ in $\Gamma(\mathcal{J}, G_K)$ defined by $f^i(y^i) = (y^i, f^i_\Delta(y^i))$ is a Nash equilibrium. Take a Nash equilibrium $e$ of $\Gamma(\mathcal{I}, G_K)$ inducing the same distribution on $S$ as $f$, and let $e^i_Y(x^i)$ and $e^i_\Delta(x^i)$ be the marginals of $e^i(x^i)$ on $Y^i$ and $\Delta(Y^{-i})$. An interpretation from $\mathcal{I}$ to $\mathcal{J}$ is again defined by $\phi^i(x^i) = e^i_Y(x^i)$. We see, as in the case where $G$ is usc, that $\phi$ is a compatible interpretation. We have to prove that $\phi$ is $\varepsilon$-faithful.

Let $P_e$ be the probability induced on $X \times Y \times \Delta(Y^{-i})$ by $\mu$ and $e$. Again, $P_\phi$ is the marginal of $P_e$ on $X \times Y$, so that $r(x^i)(y^{-i}) = P_e(y^{-i}|x^i)$. Take $x^i$, $y^i$ such that $P_\phi(x^i, y^i) > 0$. Let $U \subseteq Y^i \times \Delta(Y^{-i})$ be the support of the image of $\mu$ by $f^i$. The support $T$ of $e^i(x^i)$ is included in $U$. By definition of $f$, the section of $U$ by $\{y^i\} \times \Delta(Y^{-i})$ is $\{(y^i, f^i_\Delta(y^i))\}$. The section of $T$ by $\{y^i\} \times \Delta(Y^{-i})$ is not empty since $e^i_Y(x^i)(y^i) > 0$, therefore it is also $\{(y^i, f^i_\Delta(y^i))\}$. Then $f_\Delta(y^{-i})$ is in the support of $e^i_\Delta(x^i)$, and since $e$ is a Nash equilibrium $f^i_\Delta(y^i) \in \arg\max_\beta h^i_K(r(x^i), \beta)$. Hence $d(r(x^i), f^i_\Delta(y^i)) < \varepsilon/2$ by Lemma 10. Also, $d(q(y^i), f^i_\Delta(y^i)) < \varepsilon/2$ by definition of $f$ and by Lemma 10. Therefore $d(r(x^i), q(y^i)) < \varepsilon$. ■

*Proof of Theorem* 1. As seen in Proposition 8, the existence for all $\varepsilon > 0$ of an $\varepsilon$-faithful interpretation from $\mathcal{I}$ to $\mathcal{J}$ implies the existence of a faithful interpretation. ■

### 4.3. *Corollary*

To prove that $\mathcal{I}$ is richer than $\mathcal{J}$ if there exists a faithful transformation $\phi$ from $\mathcal{I}$ to $\mathcal{J}$, we have constructed a Nash equilibrium of $\Gamma(\mathcal{J}, G)$ from a Nash equilibrium of $\Gamma(\mathcal{I}, G)$ and from $\phi$. The following corollary shows that for this procedure to work for any $G$ and any Nash equilibrium of $\Gamma(\mathcal{I}, G)$, $\phi$ actually needs to be faithful.

COROLLARY 12. *A compatible interpretation $\phi$ from $\mathcal{I}$ to $\mathcal{J}$ is faithful if and only if for every compact game $G$ and every Nash equilibrium $f$ of $\Gamma(\mathcal{J}, G)$, the strategies defined by $e^i(x^i) = \mathbf{E}_{\phi^i(x^i)} f^i(y^i)$ form a Nash equilibrium of* $\Gamma(\mathcal{I}, G)$.

*Proof.* The direct implication is a consequence of Lemma 2 and Lemma 3. Conversely, choose $K$ and construct an equilibrium $f$ of $\Gamma(\mathcal{J}, G_K)$ as in Sec. 4.2.3. Take $x^i$, $y^i$ such that $P_\phi(x^i, y^i) > 0$. We still have $d(q(y^i), f^i_\Delta(y^i)) < \varepsilon/2$. On the other hand, $(y^i, f^i_\Delta(y^i))$ is in the support of $e^i(x^i)$. Therefore $f^i_\Delta(y^i) \in \arg\max_\beta h^i_K(r(x^i), \beta)$, and $d(r(x^i), f^i_\Delta(y^i)) < \varepsilon/2$. This finally proves that $\phi$ is $\varepsilon$-faithful for all $\varepsilon$, and therefore $\phi$ is faithful. ■

## 5. EQUIVALENCE CLASSES

In this section we present some examples and study the equivalence classes for the relation "$\mathcal{I}$ is richer than $\mathcal{J}$ and $\mathcal{J}$ is richer than $\mathcal{I}$". In particular, we exhibit minimal representatives of equivalence classes. We still write $\mathcal{I} = ((X^i)_i, \mu)$ and $\mathcal{J} = ((Y^i)_i, \nu)$, $p$ and $q$ denote the usual corresponding conditional probabilities.

### 5.1. *Equivalence relation*

DEFINITION 6. $\mathcal{I}$ and $\mathcal{J}$ are **equivalent** information structures if $\mathcal{I}$ is richer than $\mathcal{J}$ and $\mathcal{J}$ is richer than $\mathcal{I}$.

EXAMPLE 2.

| | $a_4$ | $b_4$ | $b'_4$ |
|---|---|---|---|
| $x_4$ | 1/6 | 1/18 | 1/9 |
| $x'_4$ | 1/6 | 1/18 | 1/9 |
| $y_4$ | 1/3 | 0 | 0 |

$\mathcal{I}_4$

In $\mathcal{I}_4$ the first and second rows are identical, and so are the second and third columns. If we set $\phi^1(x_4) = \phi^1(x'_4) = x_1$, $\phi^1(y_4) = y_1$, and $\phi^2(a_4) = a_1$, $\phi^2(b_4) = \phi^2(b'_4) = b_1$, we see that $\mathcal{I}_4$ is richer than $\mathcal{I}_1$. Conversely, a faithful interpretation from $\mathcal{I}_1$ to $\mathcal{I}_4$ is defined by $\phi^1(x_1) = 1/2x_4 + 1/2x'_4$, $\phi^1(y_1) = y_4$, $\phi^2(a_1) = a_4$, $\phi^2(b_1) = 1/3b_4 + 2/3b'_4$. Therefore $\mathcal{I}_4$ and $\mathcal{I}_1$ are equivalent.

### 5.2. *Minimal representatives of equivalence classes*

DEFINITION 7. An information structure $\mathcal{I}$ is **minimal** when for every $i$

$$\left\{\begin{array}{l}\mu(x^i)\mu(x'^i) > 0 \\ p(x^i) = p(x'^i)\end{array}\right. \Rightarrow x^i = x'^i.$$

EXAMPLE 3. $\mathcal{I}_1$ and $\mathcal{I}_2$ are minimal, but $\mathcal{I}_4$ is not. Nevertheless, $\mathcal{I}_4$ is equivalent to the minimal information structure $\mathcal{I}_1$.

PROPOSITION 13. *Every information structure $\mathcal{I}$ is equivalent to a minimal information structure $\tilde{\mathcal{I}}$.*

*Proof.* On $X^i$ we define an equivalence relation by $x^i \mathcal{R} x'^i$ when $p(x^i) = p(x'^i)$. The equivalence class of $x^i \in X^i$ for $\mathcal{R}$ is $\{x'^i,\ x'^i \mathcal{R} x^i\} \subseteq X^i$. Let $\tilde{X}^i \subseteq 2^{X^i}$ be the set of equivalence classes for $\mathcal{R}$, and for $x^i \in X^i$ and $\tilde{x}^i \in \tilde{X}^i$ let $\psi$ be defined by $\psi^i(x^i)(\tilde{x}^i) = 1$ if $x^i \in \tilde{x}^i$, $\psi^i(x^i)(\tilde{x}^i) = 0$ otherwise.

Let $\tilde{\mu}$ be the image by $\psi$ of $\mu$, so that $\psi$ is a compatible interpretation from $\mathcal{I}$ to $\tilde{\mathcal{I}} = ((\tilde{X}^i)_i, \tilde{\mu})$.

To get $\tilde{\mathcal{I}}$ from $\mathcal{I}$, we identified the signals of player $i$ in $\mathcal{I}$ that lead to the same conditional probability on the signals of the other players. Next lemmas show that $\mathcal{I}$ and $\tilde{\mathcal{I}}$ are equivalent, and that $\tilde{\mathcal{I}}$ is minimal. They complete the proof of Proposition 13.

LEMMA 14. *$\psi$ is a faithful interpretation from $\mathcal{I}$ to $\tilde{\mathcal{I}}$.*

*Proof.* Let $P$ be the probability induced on $X \times \tilde{X}$ by $\mu$ and $\psi$. First, see that $x^i \mathcal{R} x'^i$ implies $P(\tilde{x}^{-i}|x^i) = P(\tilde{x}^{-i}|x'^i)$. Consider $x^i \in \tilde{x}^i$ such that $\mu(x^i) > 0$. $P(\tilde{x}^{-i}|\tilde{x}^i) = \sum_{x'^i \mathcal{R} x^i} P(\tilde{x}^{-i}|x'^i)P(x'^i|\tilde{x}^i) = P(x^{-i}|x^i)$. ■

On the other hand, an interpretation $\tilde{\psi}$ from $\tilde{\mathcal{I}}$ to $\mathcal{I}$ is given by $\tilde{\psi}^i(\tilde{x}^i)(x^i) = \mu(x^i)/\tilde{\mu}(\tilde{x}^i)$ if $x^i \in \tilde{x}^i$, and $\tilde{\psi}^i(\tilde{x}^i)(x^i) = 0$ if $x^i \notin \tilde{x}^i$.

LEMMA 15. *$\tilde{\psi}$ is a faithful interpretation from $\tilde{\mathcal{I}}$ to $\mathcal{I}$.*

*Proof.* See that for $\tilde{x} \in \tilde{X}$, the restriction of $\mu$ to the Cartesian product of the sets $\tilde{x}^i$ is a product measure. If $x \in \tilde{x}$, $\mu(x) = \tilde{\mu}(x)\Pi_i \mu(x^i)/\tilde{\mu}(\tilde{x}^i)$, and thus $\mu(x) = \tilde{\mu}(\tilde{x})\tilde{\psi}(\tilde{x})(x)$. Then $\tilde{\psi}$ is a compatible interpretation from $\tilde{\mathcal{I}}$ to $\mathcal{I}$. Let $\tilde{P}$ be the probability induced by $\tilde{\mu}$ and $\tilde{\psi}$ on $\tilde{X} \times X$. For $\tilde{P}(\tilde{x}^i, x^i) > 0$, $\tilde{P}(x^{-i}|x^i) = \tilde{P}(x^{-i}|x^i, \tilde{x}^i) = \tilde{P}(x^{-i}|\tilde{x}^i)$. Therefore $\tilde{\psi}$ is faithful. ■

LEMMA 16. *$\tilde{\mathcal{I}}$ is minimal.*

*Proof.* Let $\tilde{p}(\tilde{x}^i)$ be the conditional probability of $\tilde{\mu}$ over $\tilde{X}^{-i}$ given $\tilde{x}^i$. Consider $\tilde{x}^i$ and $\tilde{x}'^i$ such that $\tilde{\mu}(\tilde{x}^i)\tilde{\mu}(\tilde{x}'^i) > 0$ and $\tilde{p}(\tilde{x}^i) = \tilde{p}(\tilde{x}'^i)$. There exists $x^i \in \tilde{x}^i$ and $x'^i \in \tilde{x}'^i$ such that $\mu(x^i)\mu(x'^i) > 0$. Since $\tilde{\psi}$ is faithful

$$p(x^i)(x^{-i}) = \mathbf{E}_{\tilde{p}(\tilde{x}^i)}\tilde{\psi}^{-i}(x^{-i}) = \mathbf{E}_{\tilde{p}(\tilde{x}'^i)}\tilde{\psi}^{-i}(x^{-i}) = p(x'^i)(x^{-i}).$$

Therefore $\tilde{x}^i = \tilde{x}'^i$. ■

Since every equivalence class contains a minimal information structure, we say that a minimal information structure is a **minimal representative** of its equivalence class.

## 5.3. *Deterministic Interpretations*

DEFINITION 8. A interpretation $\phi$ from $\mathcal{I}$ to $\mathcal{J}$ is **deterministic** when for every $i$, the support of $\phi^i(x^i)$ is a singleton $\mu$-a.s. .

EXAMPLE 4. This is the case of the previous interpretations from $\mathcal{I}_2$ and $\mathcal{I}_4$ to $\mathcal{I}_1$, but not of the one from $\mathcal{I}_1$ to $\mathcal{I}_4$.

PROPOSITION 17. *An information structure $\mathcal{J}$ is minimal if and only if for every information structure $\mathcal{I}$, any faithful interpretation from $\mathcal{I}$ to $\mathcal{J}$ is deterministic.*

*Proof.* Assume that $\mathcal{J}$ is not minimal, then we can construct $\tilde{\mathcal{J}}$ that is minimal and equivalent to $\mathcal{J}$, as well as a non-deterministic faithful interpretation $\tilde{\psi}$ from $\tilde{\mathcal{J}}$ to $\mathcal{J}$ just as in the proof of Proposition 13. Now, assume that $\mathcal{J}$ is minimal, and that $\phi$ is a faithful interpretation from $\mathcal{I}$ to $\mathcal{J}$. We need to prove that $\phi$ is deterministic. Let $P_\phi$ be the distribution induced on $X \times Y$ by $\mu$ and $\phi$. Take $x^i \in \tilde{X}^i$, and $y^i, y'^i \in Y^i$ such that $P_\phi(x^i, y^i)P_\phi(x^i, y'^i) > 0$. Since $\phi$ is faithful

$$q(y^i)(y^{-i}) = P_\phi(y^{-i}|y^i) = P_\phi(y^{-i}|x^i) = P_\phi(y^{-i}|y'^i) = q(y'^i)(y^{-i})$$

Since $\mathcal{J}$ is minimal, $y^i = y'^i$. ■

PROPOSITION 18. *If $\mathcal{I}$ and $\mathcal{J}$ are minimal and if $\mathcal{I}$ is richer than $\mathcal{J}$, then for every $i$*

$$card\ \{x^i, \mu(x^i) > 0\} \geq card\ \{y^i, \nu(y^i) > 0\}.$$

*Proof.* Consider a deterministic faithful interpretation $\phi$ from $\mathcal{I}$ to $\mathcal{J}$, then $\{y^i, \nu(y^i) > 0\}$ is the image of $\{x^i, \mu(x^i) > 0\}$ by $\phi^i$. ■

EXAMPLE 5. Since $\mathcal{I}_1$ and $\mathcal{I}_2$ are minimal, by applying Proposition 18 we see that $\mathcal{I}_1$ is not richer than $\mathcal{I}_2$.

## 6. THE CASE OF INCOMPLETE INFORMATION

To keep notations simple, we assumed complete information up to now. The model and the results extend naturally to the case of incomplete information.

We fix a (finite) set of states of nature $K$.

A **(Bayesian) game** $G = ((S^i)_i, g)$ is given by a compact set of actions $S^i$ for each player $i$ and by a continuous payoff function $g : K \times S \to \mathbb{R}^I$. As usual, $\Sigma^i = \Delta(S^i)$ is the set of mixed strategies for player $i$.

An information structure $\mathcal{I} = ((X^i), \mu)$ is now given by a (finite) set of signals $X^i$ for each player $i$ and by a probability $\mu$ over $K \times X$.

Given $\mathcal{I}$ and $G$, the extended game $\Gamma(\mathcal{I}, G)$ is the game in which:

- $(k, x)$ is drawn according to $\mu$. Each player $i$ is informed of $x^i$;
- $G$ is played.

Again, $D(\mathcal{I}, G)$ is the set of distributions on the actions of $G$ induced by Nash equilibria of $\Gamma(\mathcal{I}, G)$. We say that $\mathcal{I}$ is richer than $\mathcal{J}$ when $D(\mathcal{I}, G) \supseteq D(\mathcal{J}, G)$ for all $G$.

An interpretation $\phi$ from $\mathcal{I} = ((X^i), \mu)$ to $\mathcal{J} = ((Y^i), \nu)$ is a collection of mappings $\phi^i : X^i \to \Delta(Y^i)$ for each $i \in I$. $\phi$ and $\mu$ induce a probability $P$ on $\Delta(K \times X \times Y)$. Define $q(y^i)(k, y^{-i}) = P(k, y^{-i}|y^i)$

and $r(x^i)(k, y^{-i}) = P(k, y^{-i}|x^i)$ in $\Delta(K \times Y^{-i})$. $\phi$ is compatible when the marginal of $P$ on $K \times Y$ is $\nu$, and it is faithful when moreover, $q(y^i) = r(x^i)$ $P$-a.s. for all $i$.

Theorem 1 extends to:

THEOREM 19. *An information structure $\mathcal{I}$ is richer than another $\mathcal{J}$ if and only if there exists a faithful interpretation from $\mathcal{I}$ to $\mathcal{J}$.*

It is straightforward to see that the existence of a faithful transformation implies that $\mathcal{I}$ is richer than $\mathcal{J}$. For the other part of the proof, one can simply extend the proof of Theorem 1.

Basically, when one goes from $\mathcal{I}$ to $\mathcal{J}$ through a faithful interpretation, one removes correlation possibilities. On the other hand, a player following a faithful interpretation may not lose information on the payoff relevant state of nature $k$, nor on other player's information on $k$, nor on any higher order beliefs on $k$.

## 7. FAITHFUL INTERPRETATIONS AND STATISTICAL EXPERIMENTS

### 7.1. *Blackwell's Theorem*

Recall that an **experiment** is a collection $\alpha = (u_1, \ldots, u_n)$ of probability measures over some (finite) space $\tilde{X}$. A point $\tilde{x} \in \tilde{X}$ is selected according to one of the distributions $(u_1, \ldots, u_n)$ and is observed by the statistician. Given two experiments $\alpha$ and $\beta = (v_1, \ldots, v_n)$ with $v_k \in \Delta(\tilde{Y})$ for $k \in \{1, \ldots, n\}$, $\alpha$ is **sufficient** for $\beta$ when there exists a **stochastic transformation** from $\alpha$ to $\beta$, that is a probability transition $Q$ from $\tilde{X}$ to $\tilde{Y}$ such that the image of $u_k$ by $Q$ is $v_k$ for all $1 \le k \le n$. When $\alpha$ is sufficient for $\beta$ and $\beta$ is suficient for $\alpha$, $\alpha$ and $\beta$ are called **equivalent**. Blackwell's Theorem, (1951, 1953) states that $\alpha$ is sufficient for $\beta$ if and only if in any decision problem, the statistician guarantees a better payoff when receiving her information from $\alpha$ than from $\beta$.

To compare information structures, we followed a method similar to Blackwell's in his comparison of statistical experiments by proving the equivalence of a notion in terms of information and a notion in terms of payoffs. One may wonder if Blackwell's theorem can be seen as a particular case of ours. Consider two statistical experiments $\alpha$ and $\beta$, and let $\mathcal{I}$, $\mathcal{J}$ be two associated information structures (take for instance uniform probability on $\Omega$, $\alpha$ and $\beta$ provide transition probabilities from $\Omega$ to the sets of signals). If for any game $G$, $D(\mathcal{I}, G) \supseteq D(\mathcal{J}, G)$, then a maximizing statistician gets exactly the same payoff when getting her information from $\alpha$ or from $\beta$. Therefore $\alpha$ and $\beta$ are equivalent in Blackwell's sense whenever $\mathscr{I}$ is richer than $\mathscr{J}$. With one agent facing uncertainty on the state of nature, our comparison of information structures therefore differs from Blackwell's comparison of experiments.

### 7.2. *Alternative Definitions of a Faithful Interpretation*

Here we redefine faithful interpretations using Blackwell's comparison of experiments. Consider player $i$ as a statistician facing uncertainty on other player's signals $x^{-i}$. Two experiments $\alpha$ and $\beta$ are given by the families $\alpha^i = (u_{y^{-i}})_{\{y^{-i},\ \nu(y^{-i})>0\}}$ and $\beta^i = (v_{y^{-i}})_{\{y^{-i},\ \nu(y^{-i})>0\}}$ of probabilities over $X^i$ and $Y^i$ defined by $u_{y^{-i}}(x^i) = P_\phi(x^i|y^{-i})$ and $v_{y^{-i}}(y^i) = P_\phi(y^i|y^{-i})$. We call $\alpha^i$ the **experiment before interpretation** for player $i$, and $\beta^i$ the **experiment after interpretation** for player $i$. Since for all $y^i \in Y^i$ and $y^{-i} \in Y^{-i}$, $\mathbf{E}_{u_{y^{-i}}} \phi^i(x^i)(y^i) = \sum_{x^i} P_\phi(x^i|y^{-i})\phi^i(x^i)(y^i) = P_\phi(y^i|y^{-i})$, $\phi^i$ defines a stochastic transformation from $\alpha^i$ to $\beta^i$. Therefore $\alpha^i$ is sufficient for $\beta^i$.

THEOREM 20. *For every player i, the following statements are equivalent*:

(i) $q(y^i) = r(x^i)\ P_\phi$*-a.s.*

(ii) *The experiment before interpretation and the experiment after interpretation for player i are equivalent.*

(iii) *The distributions of* $r(x^i)$ *and* $q(y^i)$ *on* $\Delta(Y^{-i})$ *are equal.*

*In particular,* $\phi$ *is faithful if and only if any of these conditions is true for all player i.*

The proof of Theorem 20 uses a lemma.[1]

LEMMA 21. *Consider an integrable random vector z over a probability space* $(\Omega, \mathscr{A}, P)$, *and a subfield* $\mathscr{B}$ *of* $\mathscr{A}$. *Let* $\mathbf{E}_P[z|\mathscr{B}]$ *denote the conditional expectation of z on* $\mathscr{B}$. $\mathbf{E}_P[z|\mathscr{B}]$ *and z have the same distribution if and only if* $\mathbf{E}_P[z|\mathscr{B}] = z$ *P-a.s.*

*Proof of the lemma.* It is clear that $\mathbf{E}_P[z|\mathscr{B}]$ and $z$ have the same distribution if $z = \mathbf{E}_P[z|\mathscr{B}]$ $P$-a.s. Take a strictly convex application $h$ such that $h(z)$ is integrable (e.g., $h : x \to \|x\| + 1/1 + \|x\|$ with $\|x\|$ the Euclidian norm of $x$). If $\mathbf{E}_P[z|\mathscr{B}]$ and $z$ are equally distributed, both members of Jensen's inequality

$$\mathbf{E}_P h(z) \geq \mathbf{E}_P h(\mathbf{E}_P[z|\mathscr{B}])$$

are equal. Since $h$ is strictly convex, this implies $z = \mathbf{E}_P[z|\mathscr{B}]$ $P$-a.s. ∎

[1] I thank Bernard de Meyer for suggesting this lemma.

*Proof of Theorem* 20.

(i) *is equivalent to* (iii). Write $z = P_\phi(y^{-i}|x^i, y^i) = P_\phi(y^{-i}|x^i) = r(x^i)(y^{-i})$. $z$ is an integrable random vector over $(X \times Y, \mathscr{A}, P_\phi)$ with values in $\mathbb{R}^{Y^{-i}}$, where $\mathscr{A}$ is the discrete $\sigma$-algebra of $X \times Y$. Let $\mathscr{B}$ be the subfield of $\mathscr{A}$ generated by the sets $y \times x^{-i} \times X^i$ for $y \in Y$ and $x^{-i} \in X^{-i}$. Then $\mathbf{E}_{P_\phi}[z|\mathscr{B}] = P_\phi(y^{-i}|y^i) = q(y^i)(y^{-i})$, and we conclude by using Lemma 21.

(ii) *is equivalent to* (iii). In the case where the marginal of $P_\phi$ on $\{y^{-i} \in Y^{-i}, P_\phi(y^{-i}) > 0\}$ is uniform, the distributions of $r(x^i)$ and of $q(y^i)$ are the standard measures associated with $\alpha^i$ and $\beta^i$. From Theorem 4 of Blackwell, (1951), the standard measures associated with $\alpha^i$ and $\beta^i$ are equal if and only if $\alpha^i$ and $\beta^i$ are equivalent. This result easily extends to the case where the marginal of $P_\phi$ on $\{y^{-i} \in Y^{-i}, P_\phi(y^{-i}) > 0\}$ may not be uniform. ■

## 8. CONCLUDING REMARKS

We used a dual approach to the classical approach of correlated equilibria. We considered normal form games extended by information structures, but rather than keeping the game fixed and making the information structure vary to get all the correlated equilibrium distributions of the game, we compared two information structures by making the normal form game vary. We then obtained a characterization of "$\mathcal{I}$ is richer that $\mathcal{J}$" where the normal form game does not appear.

To compare two information structures, one should *a priori* check for the existence of a faithful interpretation from the one to the other. Nevertheless, it is much easier to compare minimal representatives of their equivalence classes, since any faithful interpretation from a minimal information structure to another is deterministic.

Some work remains to be done in order to connect the notions introduced here with the general theory of information structures as presented in Chapter III of Mertens-Sorin-Zamir, (1994). For instance, let $\mathcal{I}_0$ be the canonical information structure associated to $\mathcal{I}$. A consequence of Theorem 2.5 p. 148 seems to be that if $\mathcal{J}$ is richer than $\mathcal{I}$, $\mathcal{J}$ is also richer than $\mathcal{I}_0$. Since the canonical information structure associated to $\mathcal{I}_0$ is itself, this would imply that canonical information structures are the minimal elements for the preorder relation we introduced.

We assumed information structures were finite whereas we considered compact games. Note that it is always easy to prove that the existence of a faithful interpretation from an information structure to another implies the former is richer that the latter. Considering larger classes of information structures—like continous signal spaces—or smaller classes of games—like finite games—would therefore strenghten our main result.

## REFERENCES


Akerlof, G. (1970). "The Market for Lemons: Qualitative Uncertainty and the Market Mechanism," *Quart. J. Econom.* **84**, 488–500.

Aumann, R. J. (1974). "Subjectivity and Correlation in Randomized Strategies," *J. Math. Econom.* **1**, 67–95.

Aumann, R. J. (1987). "Correlated Equilibrium as an Expression of Bayesian Rationality," *Econometrica* **55**, 1–18.

Bassan, B., Scarsini, M., and Zamir S. (1997). "I Don't Want to Know!: Can it be Rational?," Discussion Paper No. 158, Center for Rationality and Interactive Decision Theory.

Blackwell, D. (1951). Comparison of experiments, in "Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability," University of California Press, 93–102.

Blackwell, D. (1953). "Equivalent Comparison of Experiments," *Ann. Math. Statist.* **24**, 265–272.

Cover, T. M., and Thomas, J. A. (1991). "Elements of Information Theory," Wiley Series in Telecommunications, Wiley, New York.

Gossner, O. (1998). "Secure Protocols—or How Communication Generates Correlation," to appear in *J. Econom. Theory* **83**, 69–89.

Green, N. L., and Stokey, J. L. (1981). "A Two-person Game of Information Transmission," Discussion Paper No. 751, Harvard Institute of Economic Research.

Hirshleifer, J. (1971). "The Private and Social Value of Information and the Reward to Inventive Activity," *Amer. Econom. Rev.* **61**, 561–574.

Lehrer, E. (1990). "Internal Correlation in Repeated Games," *International Journal of Game Theory* **19**, 431–456.

Mertens, J. F., Sorin, S., and Zamir, S. (1994). "Repeated Games," CORE Discussion Paper No. 9420-9422, Chap. III.

Monderer, D. and Samet, D. (1996). "Proximity of Information Games with Incomplete Information," *Math. Oper. Res.* **21**(3), 707–725.

Myerson, R. (1982). "Optimal Coordination Problems in Generalized Principal-Agent Problems," *J. Math. Econom.* **10**, 67–81.

Neyman, A. (1991). "The Positive Value of Information," *Games Econom. Behav.* **3**, 350–355.